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ANNIVERSARY ADDRESS

A USEFUL DEVELOPMENT IN THE CHEMISTRY OF FLAVONOIDS AS NATURAL PRODUCTS

by T. R. Seshadri, F.R.S., F.N.I., *President, National Institute of Sciences of India*

(*Delivered on 2 January 1969*)

Importance of *Ginkgo biloba* (maiden-hair tree)

Ginkgo biloba (maiden-hair tree) is a tree of great botanical and chemical interest (Fig. 1). It is the only surviving species among Ginkgoales which were widespread in the very distant past between 60 and 250 million years ago. It has apparently remained unchanged over geological ages and it has therefore been called a 'living fossil'. It has long been cultivated in temple gardens of Japan where it is considered sacred. Not only is the tree beautiful but the green fruits are eaten after frying. The flesh of the ripe fruits is found to contain toxic phenols. The tree has been introduced into European countries and the U.S.A.

The extract of the leaves of this tree (Fig. 2) has been reported to have medicinal properties and it is now being marketed in West Germany in forms suitable for oral administration and injection. It is considered to possess blood flow increasing and vascular widening properties and is recommended for use in hormonal, nervous, vascular, blood flow and nutritional disorders.

The special chemical components of these leaves have been studied intensively during the past ten years and their structures have been established. They are known as 'biflavonoids' in whose molecules two C_{15} units are linked together. In view of the important medicinal properties attributed to the leaves we have recently examined Indian plant sources for compounds of this type and have obtained some useful results.

Before proceeding further, it may be useful to mention earlier discoveries on the medicinal uses of flavonoids. These polyphenols were originally valued as mordant dyestuff and this use has become obsolete after the introduction of synthetic dyestuffs. But their importance has continued for other reasons. One of these is their vitamin P property. This was discovered by Professor Szent Georgyi more than 30 years ago. At the present time Rutin, which is obtained on a large scale from buck-wheat and eucalyptus leaves, is used in medicine. Similarly the bioflavonoids, hesperidin and naringin obtained as by-products of the citrus industry are employed for the maintenance

of capillary integrity, control of spontaneous abortion and to aid in surgery and wound healing.

Ginkgetin and related compounds

The leaves of *G. biloba* were first examined chemically by Furukawa in 1932. He proposed the structure of 5, 8-dihydroxy-4′-methoxy flavone (4′-methoxy primetin) for the chief component which he labelled as 'Compound B'. This was later on named 'ginkgetin' by Baker who also showed that it was different from synthetic—4′-methoxy primetin. The study was progressing for over 25 years without conclusive results. The main difficulties were

FIG. 1. *Ginkgo biloba.*

FIG. 2. *Ginkgo biloba*—Leaves.

as follows: (i) the leaves yielded a mixture of three closely related compounds and their separation was originally difficult; (ii) much of the earlier study was carried out with mixtures; (iii) there was also difficulty of correctly determining the molecular size of these compounds. Eventually it was concluded that they were biflavonoids containing two flavonyl units and their constitutions were arrived at by the very extensive work of Japanese and British workers.

The crude material contained at least three related natural products of which two major ones were ginkgetin and isoginkgetin. It was later shown

that these compounds could be comparatively easily isolated by taking advantage of Nakazawa's observation that ginkgetin formed a rather insoluble crystalline potassium salt. Treatment of the mixture with aqueous potassium carbonate gave the sparingly soluble potassium salt of ginkgetin and acidification of the mother liquor gave isoginkgetin. Later the surprising observation was made that the mixture could be easily separated by fractional crystallization from acetone. Since by the use of the Rast method for the determination of molecular weight satisfactory results were not obtained, the modified Menzies-Wright ebullioscopic method was used and the molecular formula of ginkgetin was found to be $C_{32}H_{22}O_{10}$ and it contained two methoxyl and four hydroxyl groups.

The UV spectra of ginkgetin tetraacetate and tetramethyl ether indicated that they are derivatives of apigenin. Moreover, the intensities of the absorption bands were approximately double those of the monomeric apigenin derivatives. This demonstrated the presence of two isolated flavonoid units per molecule. Further studies of spectra particularly IR and colour reactions gave more information about the location of groups. But definite information was obtained by chemical degradations. The most important of these are given in Chart 1.

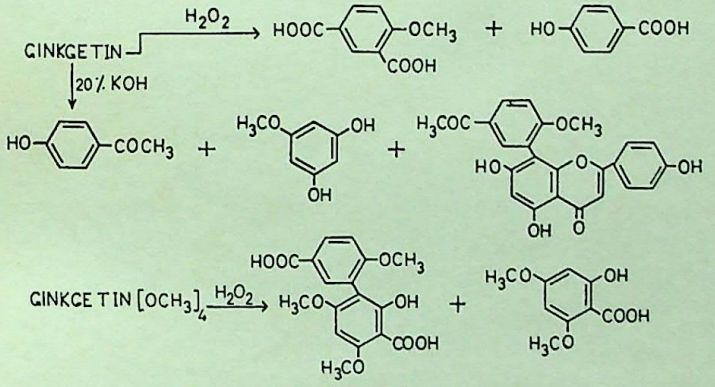


CHART 1. Chemical degradations of Ginkgetin.

They led to the conclusion that the flavonyl units had C–C linkage and were linked at 3′ and 8″-positions. The position of the methoxyls was not only indicated by the nature of the fission products but also confirmed by the study of UV spectra in the presence of bases. The constitution proposed by Baker and co-workers in the year 1959 was 4′, 7-O-dimethyl-3′, 8″-biapigeninyl (Chart 1*a*).

Nakazawa effected a synthesis of ginkgetin tetramethyl ether in 1959 by a mixed Ullmann reaction between 3′-iodo-5, 7, 4′-trimethoxyflavone and 8-iodo-5, 7, 4′-trimethoxyflavone using copper in dimethyl formamide medium. One of the products was found to be identical with the tetramethyl ether of

natural ginkgetin. In 1963, Nakazawa synthesized ginkgetin itself as follows: He used for the protection of the free hydroxyl groups in the components benzyl and benzoyl units and these were conveniently removed later on to yield ginkgetin (Chart 2).

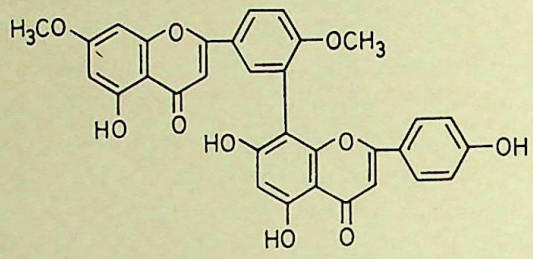


CHART 1*a*. Ginkgetin.

As already mentioned two more related compounds (partial methyl ethers) have been isolated from *G. biloba* leaves. They are isoginkgetin

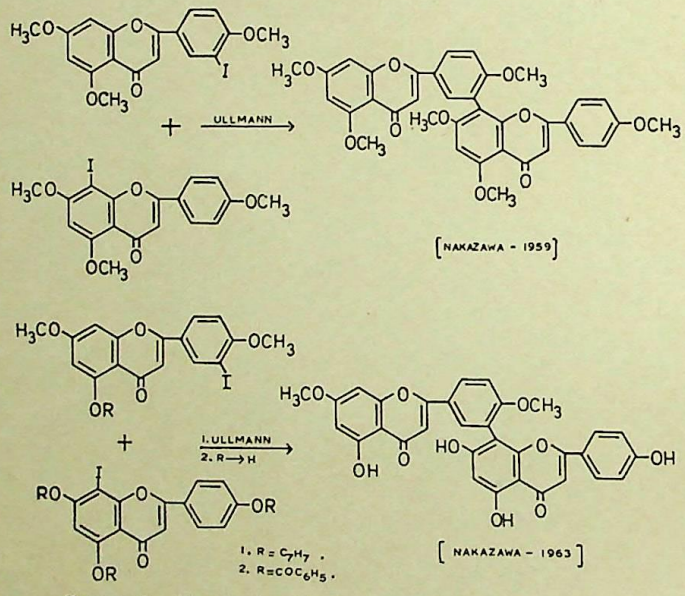
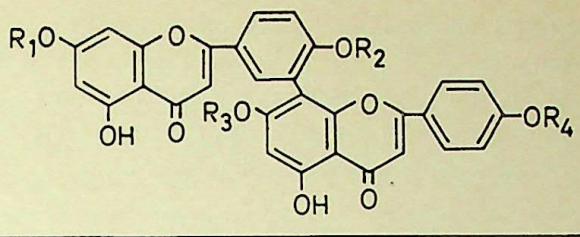


CHART 2. Synthesis of Ginkgetin and its methyl ether.

and bilobetin. The former is an isomeric dimethyl ether and the latter a related monomethyl ether. Soon many other methyl ethers of this group were isolated. The parent compound of all these ethers is called 'amentoflavone'. Originally it was considered to be rare; now several sources have been found. The trimethyl ether 'sciadopitysin' is of importance; it was isolated by Kariyone and Kawano in 1956 and their extensive work on it was complementary to the work of Baker *et al.* (1959) on ginkgetin (Chart 3).



No	Name	R_1	R_2	R_3	R_4
1.	Amentoflavone	H	H	H	H
2.	Bilobetin	H	CH_3	H	H
3.	Podocarpusflavone A	H	H	H	CH_3
4.	Sotetsuflavone	H	H	CH_3	H
5.	Ginkgetin	CH_3	CH_3	H	H
6.	Isoginkgetin	H	CH_3	H	CH_3
7.	Podocarpusflavone B	CH_3	H	H	CH_3
8.	Sciadopitysin	CH_3	CH_3	H	CH_3
9.	Kayaflavone	H	CH_3	CH_3	CH_3
10.	Amentoflavone tetramethyl ether	CH_3	CH_3	CH_3	CH_3

Chart 3. Amentoflavone series.

Hinokiflavone

A different type of biflavonoid was isolated by Kariyone and Sawada in 1958 from *Chamaecyparis obtusa*. It was named 'hinokiflavone' based on the Japanese name of the plant 'hinoki' (Chart 4). The constitution of this

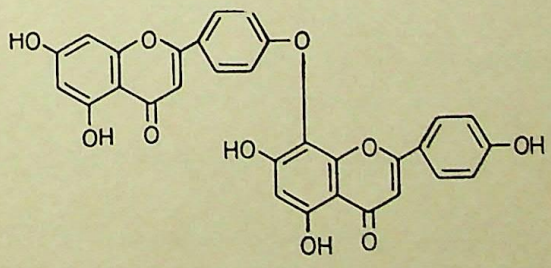


Chart 4. Hinokiflavone.

compound was arrived at by Kawano in 1960. It was also a dimer of apigenin, the molecular formula being $C_{30}H_{18}O_{10}$; it had no methoxyl groups. It formed a pentamethyl ether and pentaacetate. The UV pattern was

different from that of the earlier known biflavonyls but the UV of the complete methyl ether agreed with the UV of ginkgetin tetramethyl ether. By ferric colour reaction and IR data two hydroxyls were inferred to be in 5 and 5″ positions. From the degradations of hinokiflavone and its pentamethyl ether a diphenyl ether constitution was arrived at (Chart 5).

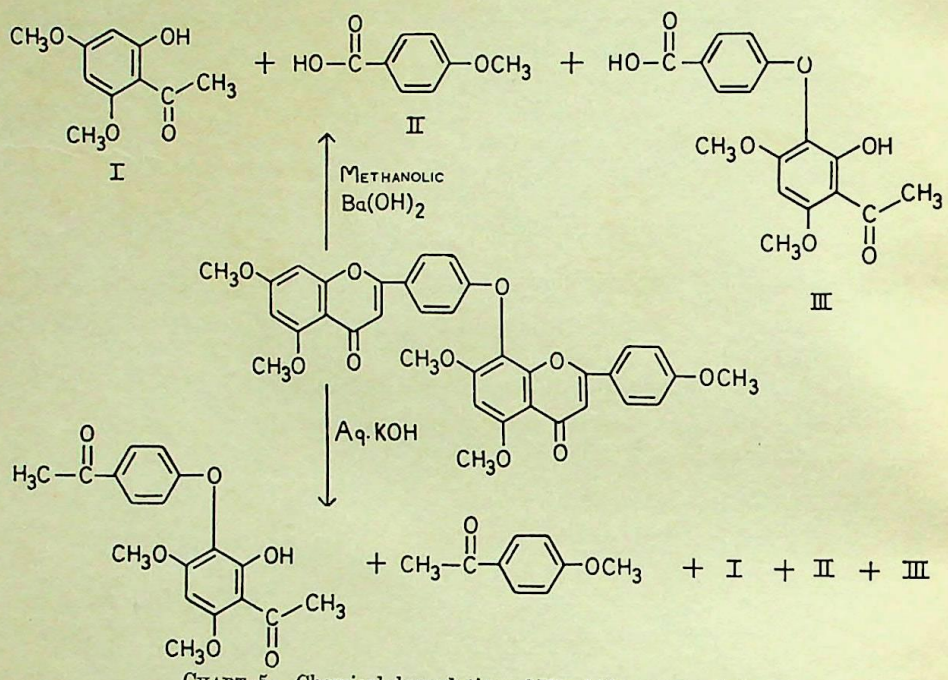


Chart 5. Chemical degradation of Hinokiflavone methyl ether.

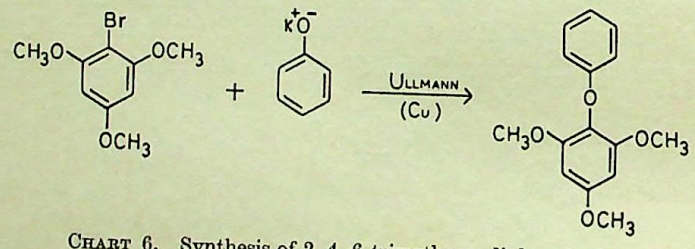


Chart 6. Synthesis of 2, 4, 6-trimethoxy diphenyl ether.

Among the products of degradation compound (III) having a diphenyl ether structure was important. It could be converted by methylation followed by oxidation and decarboxylation into 2, 4, 6-trimethoxy diphenyl ether. This could be synthesized by the Ullmann condensation of 2, 4, 6-trimethoxy-bromobenzene and potassium phenoxide. Thus the diphenyl ether structure was proved (Chart 6).

Since compound (III) gave positive reaction with Gibbs' reagent there was indication that the diphenyl ether link was ortho to the hydroxyl and the alternative possibility was eliminated. We attempted the synthesis of hinokiflavone pentamethyl ether through the Ullmann condensation of 8-iodoapigenin trimethyl ether and potassium salt of 4′-hydroxy-5, 7-dimethoxyflavone. There was partial demethylation in this reaction and remethylation was done to obtain the pentamethyl ether. The resulting product was identical with hinokiflavone pentamethyl ether. However, Nakazawa has recently claimed that he has synthesized both 4′, 8″ and 4′, 6″-isomers by unequivocal methods indicated in the formulae given below and only the 4′, 6″-compound agrees with the natural sample. This is explicable if in the above Ullmann synthesis the partial demethylation that takes place leads to isomeric change of the biflavonoid system. This is supported by his observation that synthetic 4′, 8″-biflavonyl compound gives natural hinokiflavone when boiled with hydroiodic acid. On this basis, the structure of hinokiflavone and the partial methyl ethers require revision (Chart 7).

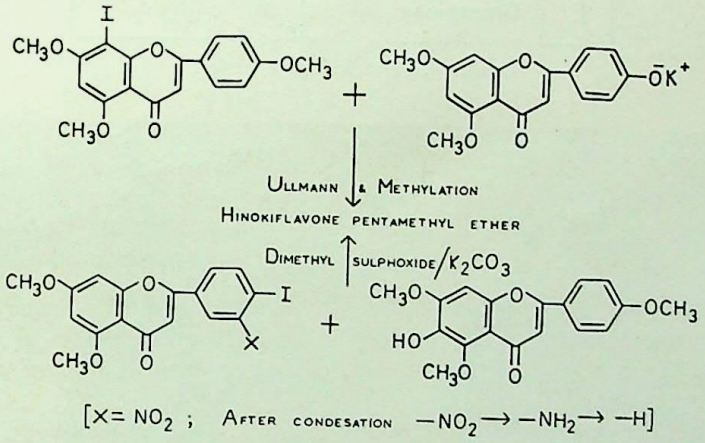


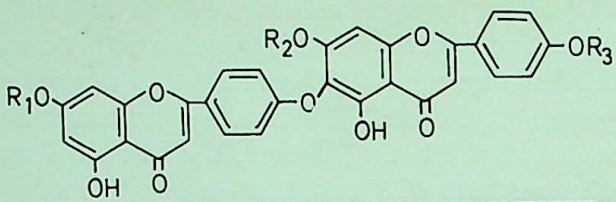
CHART 7. Synthesis of Hinokiflavone methyl ether.

Hinokiflavone is now found to occur fairly widely in nature; several of its mono and dimethyl ethers are also found (Chart 8).

Cupressuflavone

While looking for Indian sources of biflavonoids, we isolated from the leaves of *Cupressus torulosa* and *Cupressus sempervirens* (Figs. 3, 4) a new biflavone which belonged to a C–C type and called it cupressuflavone. In 1964, it was assigned the constitution of 8, 8″-biapigeninyl. Its UV spectrum was similar to that of apigenin (peaks at 226, 274 and 330 mμ). The intensity of the UV maxima suggested that cupressuflavone was a biflavone. IR spectrum indicated the presence of chelated hydroxyl (s). Its molecular

formula was found to be $C_{30}H_{18}O_{10}$. It did not contain any methoxyl or C-methyl groups. It gave a hexamethyl ether and hexaacetate. On oxidation cupressuflavone hexamethyl ether yielded anisic acid as the only



No.	Name	R_1	R_2	R_3
1.	Cryptomerin A	H	H	CH_3
2.	Isocryptomerin	H	CH_3	H
3.	Neocryptomerin	CH_3	H	H
4.	Chamaecyparin	CH_3	CH_3	H
5.	Cryptomerin B	H	CH_3	CH_3

CHART 8. Hinokiflavone series.

FIG. 3. *Cupressus torulosa.*

FIG. 4. *Cupressus sempervirens.*

isolable entity. The NMR spectrum of the complete methyl ether of cupressuflavone was symmetrical. From the coupling pattern and the chemical shifts of the aromatic protons, it was inferred that oxygenation was at 5, 7, and 4′-positions. On these considerations the following constitution was proposed (Chart 9).

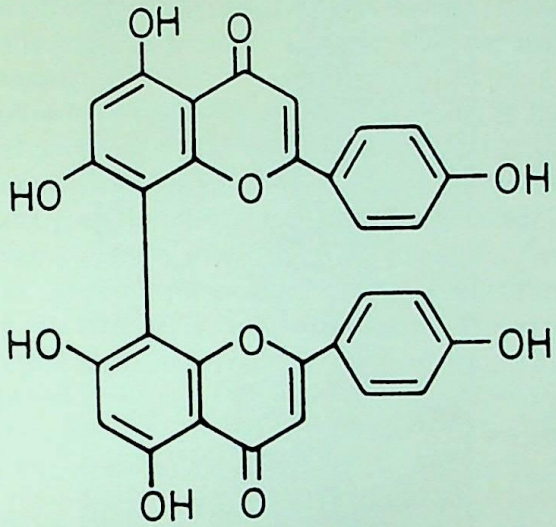


CHART 9. Cupressuflavone.

The decision between these two structures was sought to be made by the synthesis starting from 8-iodoapigenin trimethyl ether and subjecting this to Ullmann condensation. The synthetic hexamethyl ether was identical with the natural sample and from it cupressuflavone itself and its hexaacetate were obtained. But the possibility of rearrangement during these reactions has not been fully eliminated (Chart 10).

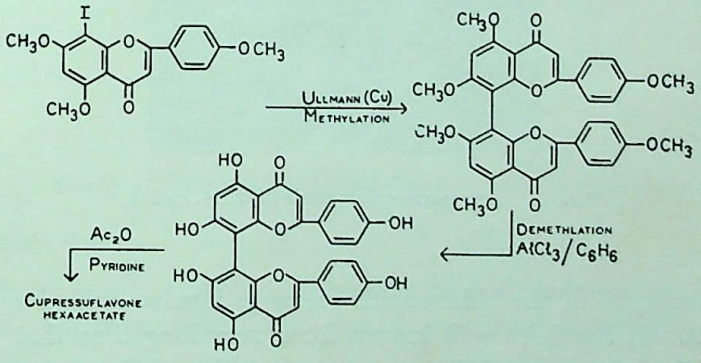


CHART 10. Synthesis of Cupressuflavone.

Recently cupressuflavone tetramethyl ether (7, 7″, 4′, 4‴-) and dimethyl ether (4′, 4‴-) were isolated by Rahman *et al.* (1968) from the leaves of

Araucaria cunnighamii and *A. cookii* (Fig. 5). These compounds are important because they exhibit optical activity arising from restricted rotation.

FIG. 5. *Araucaria columnaris* (Syn. *A. cooki*).

Biogenesis

The biogenesis of the biflavonoids seems to involve oxidative coupling of polyphenols. It has been well known for a very long time that phenols are readily oxidized by many different reagents. The products are complicated mixtures of dimeric, polymeric and quinonoid types of compounds. This method is useful for the synthesis of simple and complex compounds when coupled with efficient methods for the separation of the complex mixtures.

Phenol coupling has been suggested as an important step in the biogenesis of different classes of natural products and has stimulated intensive research. In general, the dimeric and polymeric oxidation products can be formulated by abstracting one hydrogen atom from the starting phenol and coupling together these intermediates by C–C and C–O bonds at active nuclear positions o- and p- to the hydroxyl. We owe mainly to Barton (1956) the development of these ideas leading to the synthesis of usnic acid and many others by oxidative phenol coupling.

The above considerations make it probable that the biflavonyl structure could arise by oxidative coupling of the monoflavonoid precursor apigenin. The oxygenation pattern of the precursor restricts the interflavonyl linkage to 6, 8 and 3′-nuclear positions (Chart 11). In the formation of ethers, each of these positions may be linked with any of the three oxygens of the hydroxyl groups. However, among the large number of possibilities only a few are so far known and this may be due to some factor of preferential activation. A later stage of methylation can yield the known methyl ethers. Support for this scheme is provided by the fact that in a number of plants apigenin is also present along with the biflavonyl pigments. Some experiments in our laboratory using one electron transfer reagents confirm the scheme.

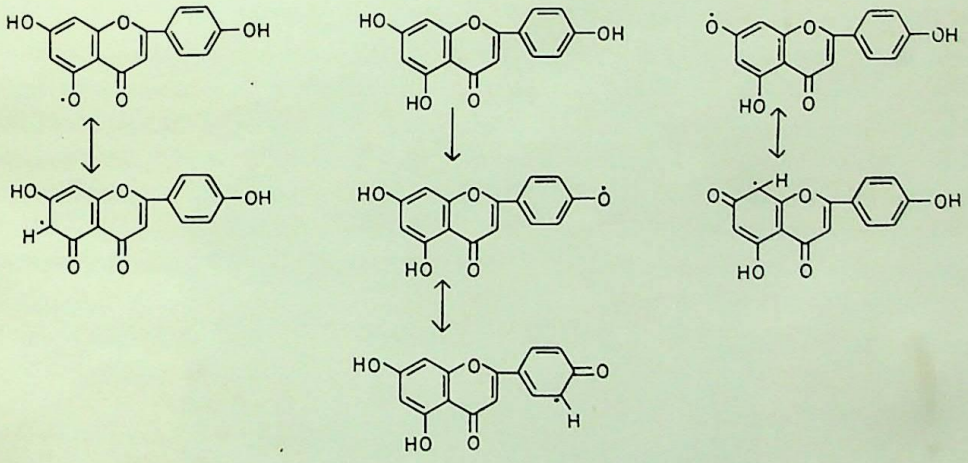


CHART 11. Free radical intermediates of oxidative coupling.

Study of Casuarinaceae

The study of the chemical components of the gymnosperms, particularly the conifers, will be useful for their possible application as drugs. There is also interest in regard to the taxonomy of the plants since they contain biflavonoids as special components. I may mention the study of *Casuarina* as a typical example. This genus has about 40 species chiefly found in Australia and in Indo-Malayan region. Of these, *Casuarina equisetifolia* is commonly

cultivated in India (Fig. 6). There is some difference of opinion regarding the taxonomic position of the genus. Sawada examined the leaves of *Casuarina stricta* and found that hinokiflavone is present in them. Earlier investigations had shown that the leaf-wax of this plant gives on hydrolysis juniperic acid (16-hydroxy palmitic acid) another typical gymnosperm constituent. The

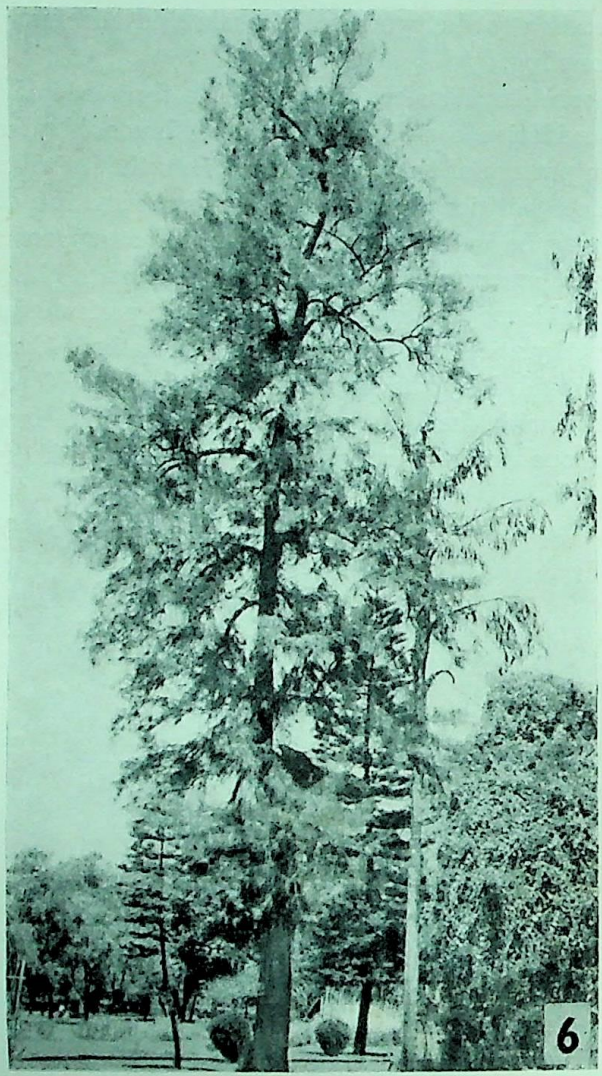


FIG. 6. *Casuarina equisetifolia.*

occurrence of these two gymnosperm compounds in the leaves of *C. stricta* lent support to the view that Casuarinaceae is close to the gymnosperms. However, Batesmith (1962) reported that *C. cunnighamii* contains only kaempferol and quercetin. Our re-examination of the leaves of Indian trees

revealed the presence of cupressuflavone. We also examined *Casuarina suberosa* and it contained hinokiflavone. These results may help to settle the taxonomic position of the genus (Chart 12).

No	Source	Biflavones Identified
	Cupressaceae	
1.	C. funebris	C, H, A, H-7″OCH$_3$.
2.	C. sempervirens	C, A.
3.	C. torulosa	C, A, H.
4.	Biota orientalis	H.
	Casuarinaceae	
1.	C. cunninghamiana	C.
2.	C. suberosa	H.

C—Cupressuflavone H-7″OCH$_3$—Isocryptomerin
A—Amentoflavone H—Hinokiflavone

Chart 12. Chemical components of some species of Cupressaceae and Casuarinaceae.

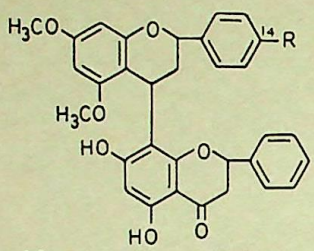
Study of mass-spectral fragmentation

During our study of the biflavonyl pigments of Cupressaceae we have found the presence of partial methyl ethers of the three known types of biflavones and also new types of biflavones. They are present in very minor amounts and chemical degradation cannot be carried out for arriving at their structures. Mass spectra can be utilized in these cases and we have therefore made a critical study of the three types of biflavones known. Since the hydroxy compounds are very high melting the spectra of their complete methyl ethers have been recorded. From such a study, we can distinguish diphenyl and diphenyl ether type of biflavones and the position of the methoxyls can also be fixed.

Other biflavonyl compounds

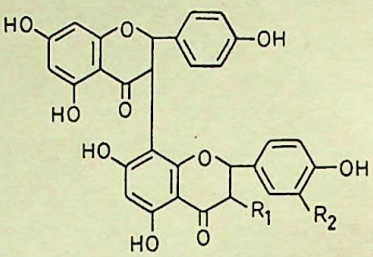
Other modes of biflavonyl linking has been shown to be possible. Recently Jackson *et al.* (1967) isolated three compounds of a new dimer that could be called flavanonyl flavanone (biflavanone) from the heartwood of *Garcinia buchanii*. This involves the linking of two flavanone units at 3 and 8 positions. Their constitutions were arrived at by UV, IR, NMR and mass spectra. In these compounds both the units consist of flavanone units. Morelloflavone, a flavonyl-flavanone, isolated by Venkataraman *et al.* (1967) from the pericarp of the seeds of *Garcinia morella* contains a linking between a flavanone and flavone. Xanthorrhone and 14-hydroxy xanthorrhone which

were isolated by Pelter *et al.* (1967) from Australian xanthorrhea contain a flavanone and flavan units (Chart 13).



XANTHORRHONE , R=H

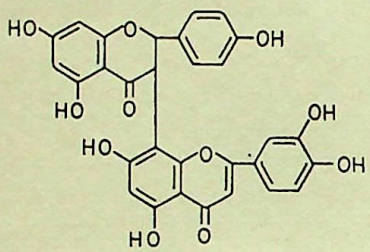
14 — HYDROXYXANTHORRHONE , R=OH



GB—1 , R_1=OH; R_2= H

GB—1A , R_1= R_2=H

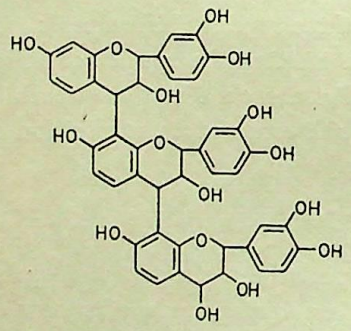
GB—2 , R_1 = R_2=OH



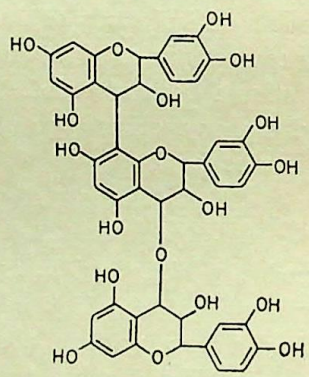
MORELLOFLAVONE

CHART 13. Recent biflavonyl compounds.

More widespread and important is another group of this type which has also been studied with some clarity during the past ten years. They are the proanthocyanidins (Chart 14). Under the name leucoanthocyanidins, they



TRI-PROFISETINIDIN



TRI-PROCYANIDIN

CHART 14. Proanthocyanidins.

had earlier attracted considerable attention as components of tanning materials and as astringent principles in foods and drugs. They are also capable of readily forming polymeric resins and a number of them have been isolated

in recent years. They are made up of either two leucoanthoanthocyanidin units or of leucoanthocyanidin and catechin. But the linking is easily broken by mineral acids, yielding eventually the corresponding anthocyanidin and catechin. Some typical examples are given in Chart 14. Frequently we obtain trimers and higher polymers.

ACKNOWLEDGEMENTS

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NOTE ON RESPONSES IN A PIEZOELECTRIC CRYSTAL WITH DIVIDED ELECTRODES

by P. Roy, *Department of Mathematics, Victoria College, Cooch Behar*

(Communicated by B. Sen, F.N.I.)

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The present note is an attempt to obtain the electrical responses in a piezoelectric crystal plate with divided electrodes with given electrical input employing the method of Laplace transform.

1. Introduction

The piezoelectric problems, particularly those concerning the electromechanical responses in a piezoelectric transducer, have long been treated from the standpoint of equivalent circuit theory, vide Mason (1948, 1950). It is only in recent years that such investigations are carried out using the principles of mechanics of continuous media and electromagnetism and the pioneering efforts in this regard have been made by Redwood (1961*a*, *b*). It is pertinent in this connection to refer here to similar attempts made by Sinha (1963, 1965, 1967*a*, *b*), Giri (1966, 1967) and Das (1967, 1969) on the determination of responses in piezoelectric crystals. The present note is an analogous attempt on a problem as yet unsolved and it seeks to work out the responses in a piezoelectric crystal with divided electrodes, some time-dependent inputs being prescribed. The analysis presented here proceeds on the lines of the mechanism of a crystal provided with electrodes as considered by Van der Veen (1956). It has been found that the Laplace transform serves as an effective tool for the solution of the problem.

2. Problem

We take a bar whose centre is chosen as the origin of coordinates (Fig. 1). The three axes X, Y, Z are in the direction of thickness ($= 2a$), the length ($= 2b$) and the width ($= 2c$). A first set of electrodes $x = \pm a$, $-b \leqslant y < 0$ and $-c \leqslant z \leqslant c$ is connected to the terminals J_1 and J_1' while a second set of electrodes with coordinates $x = \pm a$, $0 < y \leqslant b$ and $-c \leqslant z \leqslant c$ is connected to a second set of terminals J_2 and J_2'.

A longitudinal disturbance is set in piezoelectrically by applying a time-dependent voltage $V_1(t)$ across the terminals where

$$\left.\begin{aligned} V_1(t) &= 0, && t \leqslant 0 \\ &= V_0 \frac{t}{t_0}, && 0 < t \leqslant t_0 \\ &= V_0, && t > t_0 \end{aligned}\right\} \quad .. \quad .. \quad (1)$$

and the current induced I_1 is given by

$$I_1 = H(t)$$

$H(t)$ being Heaviside's steep function defined as

$$\left.\begin{aligned} H(t) &= 1, \quad t > 0 \\ &= 0, \quad t = 0 \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (1a)$$

Our object here is to find out the electrical responses represented by the voltage V_2 (say) and the current I_2 (say) across the second set of terminals arising out of the mechanical coupling between the two halves of the bar.

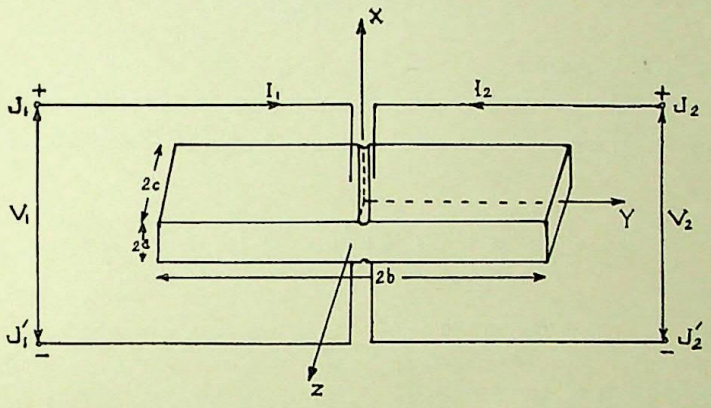

FIG. 1. Piezoelectric crystal with divided electrodes.

3. FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

The fundamental equations of the problem are evidently those which represent the interacting fields. Thus the equations are the constitutive relations of the piezoelectric material, equation of motion and the boundary conditions. The piezoelectric equations, as in Mason (1950), are given by

$$S = sT + dE \quad .. \quad .. \quad .. \quad .. \quad (2a)$$

$$D = dT + \epsilon E \quad .. \quad .. \quad .. \quad .. \quad (2b)$$

where $S = \partial\eta/\partial y$, η being the displacement in the direction of length; T, normal stress in the length direction; E, electric field strength in the thickness direction; D, electric displacement in the thickness direction; s, electric compliance; d, piezoelectric strain constant; ϵ, permittivity.

The equation of motion is

$$\frac{\partial T}{\partial y} = \rho \frac{\partial^2 \eta}{\partial t^2} \quad .. \quad .. \quad .. \quad .. \quad (3)$$

ρ being the density of the bar.

Since the electrodes are equipotential surfaces

$$\frac{\partial E}{\partial y} = 0 \quad \text{for } y \gtrless 0 \quad .. \quad .. \quad .. \quad .. \quad (4)$$

From the continuity of η and T in the plane $y = 0$, we get

$$\left.\begin{aligned}(\eta)_1 &= (\eta)_2 \\ (T)_1 &= (T)_2\end{aligned}\right\} \quad \text{on } y = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

where the suffixes indicate the respective electrodes. A third set of boundary conditions is obtained if, following Van der Veen (1956), we take into account the influence of various causes of damping of a vibrating bar by assuming a fictitious acoustical medium which exerts on the boundary surfaces ($y = \pm b$) a stress

$$(T)_{\pm b} = \mp\alpha\left(\frac{\partial\eta}{\partial t}\right)_{\pm b} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

α being a positive constant.

Again eliminating T from eqns. (2a) and (2b), integration over the first set of electrodes J_1, J_1' leads to the total charge Q on the electrodes J_1, J_1' given by

$$\begin{aligned}Q &= \frac{d}{s}\int_{-c}^{c} dz \int_{-b}^{0} S\,dy + \left(\epsilon - \frac{d^2}{s}\right)\int_{-c}^{c} dz \int_{-b}^{0} E\,dy \\ &= \phi\{\eta_{y=-0} - \eta_{y=-b}\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)\end{aligned}$$

where $\phi = 2cd/s$.

The second integral is $\frac{1}{2}C_pV_1$ where C_p is given by

$$\tfrac{1}{2}C_p = \left(\epsilon - \frac{d^2}{s}\right)\frac{cb}{a}$$

Differentiating eqn. (7) with respect to time we get the current through the terminals J_1, J_1' where

$$I_1 = \phi\frac{d}{dt}\{\eta_{-0} - \eta_{-b}\} + \frac{d}{dt}(\tfrac{1}{2}C_pV_1) \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

The second half of the crystal gives in the same way,

$$I_2 = \phi\frac{d}{dt}\{\eta_b - \eta_{+0}\} + \frac{d}{dt}(\tfrac{1}{2}C_pV_2) \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

Equations (4) to (8) constitute the boundary conditions of the problem.

4. Solution of the Problem

To solve the problem we take Laplace transform of eqns. (3)–(9) where all the quantities with bars indicate the transform of the corresponding quantity, for example,

$$\bar{\eta} = \int_0^\infty \eta e^{-pt}\,dt \quad [\mathrm{Re}\,(p) > 0]$$

Therefore, all equations mentioned above yield a second set of equations as follows:

$$\frac{\partial \bar{T}}{\partial y} = \rho p^2 \bar{\eta} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (10)$$

$$\frac{\partial \bar{\eta}}{\partial y} = 0 \qquad \text{for } y \gtrless 0 \quad .. \quad .. \quad .. \quad .. \quad (11)$$

$$(\bar{\eta})_1 = (\bar{\eta})_2 \qquad \text{for } y = 0 \quad .. \quad .. \quad .. \quad .. \quad (12)$$

$$(\bar{T})_1 = (\bar{T})_2 \qquad \text{for } y = 0 \quad .. \quad .. \quad .. \quad .. \quad (12a)$$

$$(\bar{T})_{\pm b} = \mp \alpha p (\bar{\eta})_{\pm b} \quad .. \quad .. \quad .. \quad .. \quad (13)$$

or

$$\left(\frac{\partial \bar{\eta}}{\partial y}\right)_{-b} - \alpha s p (\bar{\eta})_{-b} = \frac{d}{2a} \bar{V}_1 \quad .. \quad .. \quad .. \quad (13a)$$

$$\left(\frac{\partial \bar{\eta}}{\partial y}\right)_{b} + \alpha s p (\bar{\eta})_{b} = \frac{d}{2a} \bar{V}_2 \quad .. \quad .. \quad .. \quad .. \quad (13b)$$

$$\bar{I}_1 = \phi p \{\bar{\eta}_{-0} - \bar{\eta}_{-b}\} + \tfrac{1}{2} p C_p \bar{V}_1 \quad .. \quad .. \quad .. \quad (14)$$

and

$$\bar{I}_2 = \phi p \{\bar{\eta}_b - \bar{\eta}_{+0}\} + \tfrac{1}{2} p C_p \bar{V}_2 \quad .. \quad .. \quad .. \quad (15)$$

Solving eqn. (10) we get

$$\left.\begin{aligned} \bar{\eta} &= A e^{\frac{py}{v}} + B e^{-\frac{py}{v}} \quad \text{for } y < 0 \\ &= C e^{\frac{py}{v}} + D e^{-\frac{py}{v}} \quad \text{for } y > 0 \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (16)$$

where $\rho = 1/v^2$.

The boundary conditions (12) to (13b) yield respectively

$$A + B = C + D \quad .. \quad .. \quad .. \quad .. \quad .. \quad (17a)$$

$$A - B - C + D = \frac{dv}{2ap} (\bar{V}_1 - \bar{V}_2) \quad .. \quad .. \quad .. \quad (17b)$$

$$A e^{-\frac{pb}{v}} K_1 - B e^{\frac{pb}{v}} K_2 - \frac{d\bar{V}_1}{2ap} = 0 \quad .. \quad .. \quad .. \quad (17c)$$

$$C e^{\frac{pb}{v}} K_2 - D e^{-\frac{pb}{v}} K_1 - \frac{d\bar{V}_2}{2ap} = 0 \quad .. \quad .. \quad .. \quad (17d)$$

where

$$K_1 = 1 - \alpha v s, \quad K_2 = 1 + \alpha v s$$

From eqns. (17a)–(17d) using eqn. (14) A, B, C and D can be obtained as follows:

$$\left.\begin{aligned}
A &= \lambda_1\left\{K_2 M e^{\frac{pb}{v}} + \frac{dv\bar{V}_1}{2ap}\left(1-e^{\frac{pb}{v}}\right)\right\}\\
B &= \lambda_1\left\{K_1 M e^{-\frac{pb}{v}} - \frac{dv\bar{V}_1}{2ap}\left(1-e^{-\frac{pb}{v}}\right)\right\}\\
C &= L\lambda_2\left[M\left\{\left(1-K_1e^{-\frac{pb}{v}}\right)\left(K_2e^{\frac{pb}{v}}+K_1e^{-\frac{pb}{v}}\right)-\left(K_1e^{-\frac{pb}{v}}-K_2e^{\frac{pb}{v}}\right)\right\}\right.\\
&\quad \left.+\frac{dv}{2ap}\bar{V}_1\left\{\left(1-K_1e^{-\frac{pb}{v}}\right)\left(e^{-\frac{pb}{v}}-e^{\frac{pb}{v}}\right)-\left(e^{-\frac{pb}{v}}+e^{\frac{pb}{v}}-2\right)\right\}\right]\\
&\quad -L\frac{dv}{2ap}\bar{V}_1\\
D &= L\lambda_2\left[M\left\{\left(K_1e^{-\frac{pb}{v}}-K_2e^{\frac{pb}{v}}\right)-\left(K_2e^{\frac{pb}{v}}-1\right)\left(K_2e^{\frac{pb}{v}}+K_1e^{-\frac{pb}{v}}\right)\right\}\right.\\
&\quad \left.+\frac{dv}{2ap}\bar{V}_1\left\{e^{-\frac{pb}{v}}+e^{\frac{pb}{v}}-2-\left(K_2e^{\frac{pb}{v}}-1\right)\left(e^{-\frac{pb}{v}}-e^{\frac{pb}{v}}\right)\right\}\right]+L\frac{dv}{2ap}V_1
\end{aligned}\right\} \quad .. \quad (18)$$

where

$$\left.\begin{aligned}
L^{-1} &= 2-K_1e^{-\frac{pb}{v}}-K_2e^{\frac{pb}{v}}\\
\lambda_1^{-1} &= e^{-\frac{pb}{v}}K_1\left(1-e^{\frac{pb}{v}}\right)+K_2e^{\frac{pb}{v}}\left(1-e^{-\frac{pb}{v}}\right)\\
\lambda_2^{-1} &= K_1\left(e^{-\frac{pb}{v}}-1\right)+K_2\left(e^{\frac{pb}{v}}-1\right)
\end{aligned}\right\} \quad .. \quad .. \quad (19)$$

and

$$M = \frac{\bar{I}_1}{\phi p} - \tfrac{1}{2}C_p\frac{\bar{V}_1}{\phi}$$

and from (1) and (1a)

$$\bar{V}_1 = \frac{V_0}{p^2t_0}\left(1-e^{-pt_0}\right) \quad .. \quad .. \quad .. \quad .. \quad (20a)$$

and

$$\bar{I}_1 = \frac{1}{p} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (20b)$$

knowing the values of C and D, V_2 and $\bar{I}_2$ and hence V_2 and I_2 can be computed. Here we only compute V_2.
From eqn. (17d)

$$\bar{V}_2 = \frac{2ap}{dv}\left[Ce^{\frac{pb}{v}}K_2 - De^{-\frac{pb}{v}}K_1\right]$$

Substituting the relevant values of C and D we get $\bar{V}_2(p)$. This inversion is very complicated. Therefore, proceeding as in Redwood (1961a) we suppose

t small, i.e. p large, such that higher powers of $1/p$ can be neglected and we get on simplification

$$\begin{aligned}\bar{V}_2 &= \frac{2a}{dv}\frac{\bar{I}_1}{\phi}\left[\frac{K_2}{2(2-K_2)}-\frac{K_2}{K_1}\right]\\ &\quad -\frac{2ap}{dv}\bar{V}_1\left[\frac{K_2}{2-K_2}\left\{\frac{dv}{2ap}\left(\frac{2}{K_2}+1\right)+\frac{C_p}{\phi}\right\}\right.\\ &\quad \left.-\frac{K_1}{K_2}\left(\frac{dv}{2ap}+\tfrac{1}{2}C_pK_2\right)\right]\\ &= \frac{2a}{v\,d\phi p}\left[\frac{K_2}{2(2-K_2)}-\frac{K_1}{K_2}\right]\\ &\quad -\frac{2a}{dv}\frac{V_0}{pt_0}(1-e^{-pt_0})\left[\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2}-\frac{K_1}{K_2}\right\}\frac{dv}{2ap}\right.\\ &\quad \left.+\left\{\frac{K_2}{(2-K_2)\phi}-\frac{K_1}{2}\right\}C_p\right]\\ &= \frac{2a}{dv}\frac{1}{\phi p}\left[\frac{K_2}{2(2-K_2)}-\frac{K_1}{K_2}\right]\\ &\quad -\frac{2a}{dv}\frac{V_0}{t_0}\left[\frac{dv}{2a}\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2}-\frac{K_1}{K_2}\right\}\frac{(1-e^{-pt_0})}{p^2}\right.\\ &\quad \left.+C_p\left\{\frac{K_2}{(2-K_2)\phi_2}-\frac{K_1}{2}\right\}\frac{(1-e^{-pt_0})}{p}\right]\end{aligned}$$

Inverting this we have, vide Churchill (1958),

$$\begin{aligned}V_2(t) &= \frac{2a}{dv\phi}\left[\frac{K_1}{2(2-K_2)}-\frac{K_1}{K_2}\right]H(t)\\ &\quad -\frac{2a}{dv}\frac{V_0}{t_0}\left[\frac{dv}{2a}\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2}-\frac{K_1}{K_2}\right\}t_0t\right.\\ &\quad \left.+C_p\left\{\frac{K_2}{(2-K_2)\phi}-\frac{K_2}{2}\right\}\right] \quad \text{for } 0<t\leqslant t_0\\ &= \frac{2a}{dv\phi}\left[\frac{K_1}{2(2-K_2)}-\frac{K_1}{K_2}\right]H(t)\\ &\quad -\frac{2a}{dv}\frac{V_0}{t_0}\left[\frac{dv}{2a}\left\{\left(\frac{2}{K_2}+1\right)\frac{K_2}{2-K_2}-\frac{K_1}{K_2}\right\}\right]t_0 \quad \text{for } t>0\end{aligned}$$

Thus we find that the electrical voltage response has different characteristics for different ranges of time, namely part of it is step in character throughout while another part of it is linear in $0<t\leqslant t_0$ and this latter part continues to be constant beyond time t_0.

This expression $V_2(t)$ enables us to calculate $I_2(t)$ from eqn. (15).

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References

Churchill, R. V. (1958). Operational Mathematics, second edition. McGraw-Hill Book Co. Inc., New York.

Das, N. C. (1967). On electrical response in a piezoelectric plate transducer with a prescribed input. *Indian J. Phys.*, **42**, 611.

——— (1969). On mechanical response of a composite piezoelectric plate transducer characterized by a time-decaying polarization gradient. *Proc. natn. Inst. Sci. India*, **35** A (Suppl. 2), 173.

Giri, R. R. (1966). Note on the study of responses in a piezoelectric plate transducer when the input is partly periodic and partly periodic transient. *Revue roum. Sci. Techn.*, **11**, 253.

——— (1967). Note on the responses of a composite piezoelectric transducer subjected to a constant heat flow and to a time-dependent electrical voltage. *Proc. natn. Inst. Sci. India*, **33** A, 325.

Mason, W. P. (1948). Electromechanical Transducers and Wave Filters, second edition. D. Van Nostrand Co. Inc., New York.

——— (1950). Piezoelectrical Crystals and Their Applications to Ultrasonics. D. Van Nostrand Co. Inc., New York.

Redwood, M. (1961*a*). Transient performance of a piezoelectrical transducer. *J. acoust. Soc. Am.*, **33**, 527.

——— (1961*b*). Piezoelectric generation of an electrical impulse. *J. acoust. Soc. Am.*, 1386.

Sinha, D. K. (1963). Note on electrical and mechanical responses with ramp-type input signal in piezoelectric plate transducer. *Indian J. theor. Phys.*, **11**, 93.

——— (1965). A note on mechanical response in a piezoelectric transducer owing to an impulsive voltage input. *Proc. natn. Inst. Sci. India*, **31** A, 395.

——— (1967*a*). Mechanical response in a piezoelectric transducer subjected to a constant flow of heat and step-voltage input. *Proc. Indian Acad. Sci.*, **65**, 136.

——— (1967*b*). A note on response in a piezoelectric transducer. *Indian J. pure appl. Phys.*, **5** (8), 375.

Van der Veen, B. (1956). The equivalent network of a piezoelectric crystal with divided electrodes. *Phillips. Res. Rep.*, **11**, 66.

SOME SPECIAL CASES OF SPHERICALLY SYMMETRIC GRAVITATIONAL COLLAPSE WITH REFERENCE TO NEUTRINO EMISSION

by K. P. Singh *and* M. C. Srivastava, *Department of Mathematics, Banaras Hindu University, Varanasi*

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Misner's field equations (1965) for spherically symmetric gravitational collapse with escaping neutrinos have been considered in some particular cases. It is shown that spherically symmetric gravitational collapse with neutrino emission is not possible if the congruence associated with the matter field satisfies any two of the following properties: (i) it is geodetic, (ii) it is expansion-free and (iii) it is shear-free. Also the field does not permit neutrino emission if the null congruence associated with neutrino emission is expansion-free. A case has been worked out with geodetic matter congruence where the possibility of neutrino emission is demonstrated. Similarly it is shown that neutrino emission is possible in the case of shear-free matter congruence.

1. Introduction

It has been suggested by Colgate and White (Misner 1965) and also by Chiu (1964) that in case of gravitational collapse of supernovae and quasi-stellar radio sources the emission of neutrinos is possible. Misner (1965) formulated the field equations for a spherically symmetric case of gravitational collapse with escaping neutrinos. It has been assumed that the neutrinos after emission travel in the radial direction. In the absence of neutrino emission the field reduces to that obtained by Misner and Sharp (Misner 1965). The internal field of collapsing matter is continued with the field of pure radiation such as the one obtained by Vaidya (1953). In this paper we discuss some of the special features of the field equations of spherical gravitational collapse with reference to the possibility of neutrino emission. It is shown that spherically symmetric gravitational collapse with neutrino emission is not possible if the congruence associated with the matter field satisfies any two of the following properties: (i) it is geodetic, (ii) it is expansion-free and (iii) it is shear-free. Also the field does not permit neutrino emission if the null congruence associated with neutrino is expansion-free. A case has been worked out with geodetic matter congruence where the possibility of neutrino emission is demonstrated. Similarly it is shown that neutrino emission is possible in the case of shear-free matter congruence.

2. The Field Equations, Kinematical Quantities and Optical Parameters

We give below the system of field equations and define the various symbols involved. The meaning of the symbols is explained subsequently.

The field equations are as follows (Misner 1965):

$$A \equiv (nu^a)_{;\,a} = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.1)$$

$$B \equiv -nc + u^a T^b_{a;\,b} = 0 \quad .. \quad .. \quad .. \quad .. \quad (2.2)$$

$$D^a_b \equiv R^a_b - \tfrac{1}{2}\delta^a_b R + 8\pi(T^a_b + N^a_b) = 0 \quad .. \quad .. \quad .. \quad (2.3)$$

where

$$E \equiv u^a u_a + 1 = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.4)$$

$$T^a_b = (\epsilon + p)u^a u_b + \delta^a_b p \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.5)$$

$$N^a_b = qk^a k_b \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.6)$$

and

$$F \equiv k^a k_a = 0. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.7)$$

Next we consider the following kinematical quantities associated with u^a:

$$v_I \equiv u^a_{;\,a} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.8)$$

$$\omega^a \equiv \tfrac{1}{2}\eta^{abcd} u_b u_{c;\,d} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.9)$$

$$-\omega_{ab} \equiv \tfrac{1}{2}(u_{a;\,b} - u_{b;\,a}) + \tfrac{1}{2}(\dot{u}_a u_b - \dot{u}_b u_a) \quad .. \quad .. \quad .. \quad .. \quad (2.10)$$

$$\sigma_{ab} \equiv \tfrac{1}{2}(u_{a;\,b} + u_{b;\,a}) + \tfrac{1}{2}(\dot{u}_a u_b + \dot{u}_b u_a) - \frac{v_I}{3}(g_{ab} + u_a u_b) \quad .. \quad (2.11)$$

where

$$\dot{u}_a \equiv u_{a;\,b} u^b \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.12)$$

and η^{abcd} is the pseudo tensor density with $\eta^{1234} = \dfrac{1}{\sqrt{-g}}$.

Also the 'optical parameters' are

$$v_{II} \equiv \tfrac{1}{2}k^a_{;\,a} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.13)$$

$$\omega \equiv \{\tfrac{1}{4}(k_{a;\,b} - k_{b;\,a})k^{a;\,b}\}^{\frac{1}{2}} \quad .. \quad .. \quad .. \quad .. \quad (2.14)$$

$$\Sigma \equiv \tfrac{1}{2}k_{a;\,b}\bar{l}^a \bar{l}^b \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.15)$$

$$\Omega \equiv \tfrac{1}{2}k_{a;\,b}\bar{l}^a m^b \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.16)$$

where

$$G^a \equiv k^a_{;\,b} k^b = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.17)$$

$$l^a l_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.18)$$

$$\bar{l}^a \bar{l}_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.19)$$

$$m^a m_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.20)$$

$$l^a k_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.21)$$

$$\bar{l}^a k_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.22)$$

$$l^a m_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.23)$$

$$\bar{l}^a m_a = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.24)$$

$$l^a \bar{l}_a - 1 = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.25)$$

$$k^a m_a - 1 = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.26)$$

and

$$H_{ab} \equiv g_{ab} - (l_a \bar{l}_b + l_b \bar{l}_a) - (k_a m_b + k_b m_a) = 0. \quad .. \quad .. \quad (2.27)$$

Equation (2.1) is the 'equation of continuity' for matter, eqn. (2.2) correlates the cooling rate of unit volume of matter with the rate of decrease of internal energy due to neutrino emission and eqn. (2.3) represents the space-time structure in terms of total stress-energy. Here n is the baryon number density and c the cooling rate of unit amount of matter. The assumption of perfect fluid distribution form of the stress-energy tensor T_{ab} for matter is inherent in (2.5) while (2.6) gives the 'geometrical optics' form of the stress-energy tensor N_{ab} for neutrinos. Here ϵ, p and q are respectively the matter density, pressure and neutrino energy density.

The kinematical quantities (Witten 1962) for an observer following one of the curves of the time-like congruence and using Fermi-propagated axes are described as follows: v_I, ω_{ab}, σ_{ab} are respectively the velocity of expansion, rotation and shear of the neighbouring cloud of particles. The vector ω^a is the angular velocity in the infinitesimal rest-space of the observer. The 'optical parameters' (Witten 1962, p. 58) v_{II}, ω, Σ and Ω are called the expansion, twist, shear and rotation of the null congruence with respect to the observer. A semi-colon (;) preceding a suffix indicates covariant derivative. Here we have 37 equations:

$$1(A)+1(B)+10(D^a_b)+1(E)+1(F)+4(G^a)+9\{\text{eqns. (2.18)–(2.26)}\}+10(H_{ab})$$

in 35 unknowns:

$$1(n)+4(u^a)+1(c)+3(\epsilon, p, q)+16(l^a, \bar{l}^a, k^a, m^a)+10(g_{ab}).$$

3. The Case of Spherical Symmetry

We consider the spherically symmetric metric

$$ds^2 = -e^{2\alpha}\,dt^2 + e^{\beta}\,dr^2 + \gamma^2(d\theta^2 + \sin^2\theta\,d\phi^2) \quad .. \quad .. \quad (3.1)$$

where α, β, γ are functions of r and t. The coordinates r, θ, ϕ and t will be denoted by x^1, x^2, x^3 and x^4 respectively.

$$u^a = 0 \quad \text{for } a = 1, 2, 3; \quad u^4 = e^{-\alpha}. \quad .. \quad .. \quad (3.2)$$

Next we choose the orthonormal tetrad given by

$$\{\lambda^a_{b|}\} \equiv \text{diag}\{\sqrt{g^{11}}, \sqrt{g^{22}}, \sqrt{g^{33}}, \sqrt{-g^{44}}\}$$

and pseudo-orthonormal tetrad $(l^a, \bar{l}^a, k^a, m^a)$ is defined as

$$\left.\begin{aligned} l^a &= \frac{1}{\sqrt{2}}(\lambda^a_{2|}+i\lambda^a_{3|}), \quad \bar{l}^a = \frac{1}{\sqrt{2}}(\lambda^a_{2|}-i\lambda^a_{3|}) \\ k^a &= \frac{1}{\sqrt{2}}(\lambda^a_{1|}+\lambda^a_{4|}), \quad m^a = \frac{1}{\sqrt{2}}(\lambda^a_{1|}-\lambda^a_{4|}) \end{aligned}\right\} \quad .. \quad .. \quad (3.3)$$

where a suffix followed by a vertical stroke indicates the particular vector. Equation (2.27) can equivalently be written as

$$H_{ab} \equiv g_{ab} - \sum_{\nu=1,2,3} \lambda_{\nu|a}\lambda_{\nu|b} + \lambda_{4|a}\lambda_{4|b} = 0.$$

For the metric (3.1) the non-vanishing components of $(l^a, \bar{l}^a, k^a, m^a)$ are

$$\left.\begin{aligned} l_2 = l_3/i \sin\theta = \bar{l}_2 = -\bar{l}_3/i \sin\theta &= \frac{\gamma}{\sqrt{2}} \\ k_1 = m_1 = \frac{e^{\beta/2}}{\sqrt{2}}, \quad k_4 = -m_4 = -\frac{e^{\alpha}}{\sqrt{2}} & \end{aligned}\right\} \quad .. \quad .. \quad (3.4)$$

and due to (3.2) and (3.4) eqns. (2.1) and (2.2) reduce to

$$n = F(r)\frac{e^{-\beta/2}}{\gamma^2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.5)$$

$$c = e^{-\alpha}(\epsilon+p)\frac{n_{,4}}{n} - e^{-\alpha}\frac{\epsilon_{,4}}{n} \quad .. \quad .. \quad .. \quad (3.6)$$

where $F(r)$ is an arbitrary function and $n_{,4} \equiv \dfrac{\partial n}{\partial t}$.

The non-vanishing components of D_{ab} in (2.3) are

$$D_{14} \equiv -8\pi\frac{q}{2} + 2e^{-\alpha-\beta/2}\left(\frac{\gamma_{14}}{\gamma} - \frac{\gamma_1}{\gamma}\frac{\beta_4}{2} - \frac{\gamma_4}{\gamma}\alpha_1\right) = 0 \quad .. \quad .. \quad .. \quad (3.7)$$

$$D^1_1 \equiv -8\pi\left(p+\frac{q}{2}\right) - e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma} - \frac{2\gamma_4}{\gamma}\alpha_4 + \frac{\gamma_4^2}{\gamma^2}\right) + e^{-\beta}\left(\frac{\gamma_1^2}{\gamma^2} + \frac{2\gamma_1}{\gamma}\alpha_1\right) - \frac{1}{\gamma^2} = 0$$

$$.. \quad (3.8)$$

$$D^2_2 = D^3_3 \equiv -8\pi p - e^{-2\alpha}\left(\frac{\gamma_{44}}{\gamma} - \frac{\gamma_4}{\gamma}\alpha_4 + \frac{\gamma_4}{\gamma}\frac{\beta_4}{2} + \frac{\beta_{44}}{2} + \frac{\beta_4^2}{4} - \alpha_4\frac{\beta_4}{2}\right)$$

$$-e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma} + \frac{\gamma_1}{\gamma}\frac{\beta_1}{2} - \frac{\gamma_1}{\gamma}\alpha_1 + \alpha_1\frac{\beta_1}{2} - \alpha_1^2 - \alpha_{11}\right) = 0 \quad .. \quad (3.9)$$

$$D^4_4 \equiv 8\pi\left(\epsilon+\frac{q}{2}\right) - e^{-2\alpha}\left(\frac{\gamma_4^2}{\gamma^2} + \frac{\gamma_4}{\gamma}\beta_4\right) - e^{-\beta}\left(-\frac{2\gamma_{11}}{\gamma} - \frac{\gamma_1^2}{\gamma^2} + \frac{\gamma_1}{\gamma}\beta_1\right) - \frac{1}{\gamma^2} = 0$$

where $$.. \quad (3.10)$$

$$\gamma_1 \equiv \frac{\partial\gamma}{\partial r}, \quad \gamma_4 \equiv \frac{\partial\gamma}{\partial t}, \text{ etc.}$$

Equation (2.17) reduces to

$$e^{\beta/2}\frac{\beta_4}{2} + e^{\alpha}\alpha_1 = 0. \quad .. \quad .. \quad .. \quad .. \quad (3.11)$$

Here we observe that for spherical symmetry ω^a, ω_{ab} and Σ vanish identically, whereas ω and Ω vanish due to (3.11). Consequently, the congruences associated with u^a and k^a are necessarily normal. The non-vanishing kinematical quantities and parameters are given by

$$v_I = e^{-\alpha}\left(\frac{\beta_4}{2}+\frac{2\gamma_4}{\gamma}\right) \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.12)$$

$$\sigma_{11} = -\frac{2e^{\beta}}{\gamma^2}\sigma_{22} = -\frac{2e^{\beta}}{\gamma^2}\frac{\sigma_{33}}{\sin^2\theta} = \frac{e^{\beta-\alpha}}{3}\left(\beta_4-\frac{2\gamma_4}{\gamma}\right) \quad .. \quad .. \quad (3.13)$$

$$\dot{u}_1 = \alpha_1, \quad \dot{u}_2 = \dot{u}_3 = \dot{u}_4 = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.14)$$

$$v_{II} = \frac{1}{\sqrt{2}}\left(e^{-\beta/2}\frac{\gamma_1}{\gamma}+e^{-\alpha}\frac{\gamma_4}{\gamma}\right). \quad .. \quad .. \quad .. \quad .. \quad (3.15)$$

Thus we have seven equations, viz. (3.5) to (3.11) in eight unknowns, viz. n, c, ϵ, p, q, α, β, γ, along with v_I, v_{II}, σ_{11}, $\dot{u}_1$. In the next section we study the consequences when some of the quantities v_I, v_{II}, σ_{11} and $\dot{u}_1$ vanish.

4. Special Cases

The eqns. (3.7), (3.8), (3.9) imply

$$-2e^{-\alpha-\beta/2}\left(\frac{\gamma_{14}}{\gamma}-\frac{\gamma_1}{\gamma}\frac{\beta_4}{2}-\frac{\gamma_4}{\gamma}\alpha_1\right)+e^{-2\alpha}\left(-\frac{\gamma_{44}}{\gamma}+\frac{\gamma_4}{\gamma}\alpha_4+\frac{\gamma_4}{\gamma}\frac{\beta_4}{2}-\frac{\gamma_4^2}{\gamma^2}+\frac{\beta_{44}}{2}+\frac{\beta_4^2}{4}-\alpha_4\frac{\beta_4}{3}\right)$$

$$+e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma}+\frac{\gamma_1}{\gamma}\frac{\beta_1}{2}+\frac{\gamma_1}{\gamma}\alpha_1+\frac{\gamma_1^2}{\gamma^2}+\alpha_1\frac{\beta_1}{2}-\alpha_1^2-\alpha_{11}\right)-\frac{1}{\gamma^2}=0. \quad (4.1)$$

Now we consider the following cases:

Case (i)—$v_I = \sigma_{11} = 0$.

Equations (3.12) and (3.13) give

$$\frac{\beta_4}{2}+\frac{2\gamma_4}{\gamma}=0$$

$$\beta_4-\frac{2\gamma_4}{\gamma}=0$$

which imply

$$\beta_4=\gamma_4=0$$

and hence from (3.7) we get

$$q=0.$$

Since $e^{-\alpha}\gamma_4$ is the velocity of the fluid surface, $\gamma_4 = 0$ implies no collapse while $q = 0$ implies absence of neutrinos. Hence expansion-free and shear-free (or rigid) congruences associated with matter field imply no spherical gravitational collapse with escaping neutrinos.

Case (ii)—$\dot{u}_1 = \sigma_{11} = 0$.

Equations (3.14), (3.13), (3.11) and (3.7) imply

$$\alpha_1=\beta_4=\gamma_4=q=0.$$

Hence geodetic shear-free congruences associated with matter field do not permit spherical gravitational collapse with escaping neutrinos.

Combining cases (i) and (ii) with $\omega^a \equiv 0$ we have:

Spherical gravitational collapse with escaping neutrinos does not admit a space-time which is the direct product of a time-like line and a 3-space (Witten 1962, p. 58).

Case (iii)—$\dot{u}_1 = v_I = 0$.

Equations (3.14), (3.12), (3.11) and (3.7) imply

$$\alpha_1 = \beta_4 = \gamma_4 = q = 0.$$

Hence geodetic expansion-free congruences associated with matter field imply no spherical gravitational collapse with escaping neutrinos.

Case (iv)—$v_{II} = 0$.

Equation (3.15) gives

$$e^{-\beta/2}\gamma_1 + e^{-\alpha}\gamma_4 = 0. \quad .. \quad .. \quad .. \quad .. \quad (4.2)$$

From (4.1), (4.2) and (3.9) we get

$$e^{-2\alpha}\left(\frac{\beta_{44}}{2} - \alpha_4\frac{\beta_4}{2}\right) + e^{-\beta}\left(\alpha_1\frac{\beta_1}{2} - \beta_{11}\right) = \frac{1}{\gamma^2} \quad .. \quad .. \quad (4.3)$$

and

$$-8\pi p = e^{-2\alpha}\left(\frac{\gamma_{44}}{\gamma} - \frac{\gamma_4}{\gamma}\alpha_4\right) + e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma} + \frac{\gamma_1}{\gamma}\frac{\beta_1}{2}\right) + \frac{1}{\gamma^2}. \quad .. \quad (4.4)$$

Equations (4.2), (3.7) and (3.10) give

$$8\pi\epsilon = e^{-2\alpha}\left(\frac{\gamma_{44}}{\gamma} - \frac{\gamma_4}{\gamma}\alpha_4\right) + e^{-\beta}\left(-\frac{\gamma_{11}}{\gamma} + \frac{\gamma_1}{\gamma}\frac{\beta_1}{2}\right) + \frac{1}{\gamma^2} \quad .. \quad (4.5)$$

which along with (4.4) implies

$$\epsilon = -p.$$

Since ϵ and p both should be positive we have

$$\epsilon = p = 0;$$

then field equations (3.7) and (3.8) give

$$-2e^{-\alpha-\beta/2}\left(\frac{\gamma_{14}}{\gamma} - \frac{\gamma_1}{\gamma}\frac{\beta_4}{2} - \frac{\gamma_4}{\gamma}\alpha_1\right) = e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma} - \frac{2\gamma_4}{\gamma}\alpha_4\right) - 2e^{-\beta}\frac{\gamma_1}{\gamma}\alpha_1 + \frac{1}{\gamma^2}$$

which by virtue of (4.2) reduces to

$$\frac{1}{\gamma^2} = 0.$$

This is not compatible with the metric. Hence geodetic expansion-free null congruences associated with neutrino emission imply no spherical gravitational collapse with escaping neutrinos.

Case (*v*)—$\dot{u}_1 = 0$.

Equation (3.14) gives

$$\alpha_1 = 0$$

and (3.11) gives

$$\beta_4 = 0.$$

Then (3.1) can be written as

$$ds^2 = -f(t)\,dt^2 + g(r)\,dr^2 + \gamma^2(r, t)(d\theta^2 + \sin^2\theta\,d\phi^2)$$

which can be transformed to

$$ds^2 = -dt^2 + dr^2 + \gamma^2(d\theta^2 + \sin^2\theta\,d\phi^2). \qquad .. \quad .. \quad (4.6)$$

For this metric we have

$$u^1 = u^2 = u^3 = 0, \quad u^4 = 1$$

$$l_1 = l_4 = \bar{l}_1 = \bar{l}_4 = k_2 = k_3 = m_2 = m_3 = 0$$

$$l_2 = \frac{l_3}{i\sin\theta} = \bar{l}_2 = -\frac{\bar{l}_3}{i\sin\theta} = \frac{\gamma}{\sqrt{2}}, \quad k_1 = -k_4 = m_1 = m_4 = \frac{1}{\sqrt{2}}$$

$$n = \frac{F(r)}{\gamma^2}, \quad c = \frac{1}{4\pi F}\{\gamma_{14}(\gamma_1+\gamma_4)+\gamma(\gamma_{114}+\gamma_{144})\}$$

$$2\pi q = \frac{\gamma_{14}}{\gamma} \qquad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4.6a)$$

$$-8\pi\left(p+\frac{q}{2}\right) = \frac{2\gamma_{44}}{\gamma}+\frac{\gamma_4^2}{\gamma^2}-\frac{\gamma_1^2}{\gamma^2}+\frac{1}{\gamma^2} \qquad .. \quad .. \quad .. \quad .. \quad (4.6b)$$

$$-8\pi p = \frac{\gamma_{44}}{\gamma}-\frac{\gamma_{11}}{\gamma} \qquad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4.6c)$$

$$8\pi\left(\epsilon+\frac{q}{2}\right) = -\frac{2\gamma_{11}}{\gamma}-\frac{\gamma_1^2}{\gamma^2}+\frac{\gamma_4^2}{\gamma^2}+\frac{1}{\gamma^2} \qquad .. \quad .. \quad .. \quad .. \quad (4.6d)$$

$$v_I = 2\gamma_4/\gamma$$

$$\sigma_{11} = -2\sigma_{22}/\gamma^2 = -2\sigma_{33}/\gamma^2\sin^2\theta = -2\gamma_4/3\gamma$$

$$v_{II} = \frac{1}{\sqrt{2}}\left(\frac{\gamma_1}{\gamma}+\frac{\gamma_4}{\gamma}\right)$$

and (4.1) reduces to

$$\gamma(\gamma_{11}+2\gamma_{14}+\gamma_{44})+\gamma_4^2-\gamma_1^2+1 = 0. \qquad .. \quad .. \quad (4.7)$$

One solution of (4.7) for which neutrino emission is possible is

$$\gamma = f(lr+mt)$$

where l, m are constants and function f satisfies the equation

$$(l+m)^2ff''+(m^2-l^2)f'^2+1 = 0 \qquad .. \quad .. \quad .. \quad (4.8)$$

where

$$f' \equiv \frac{df}{du}, \quad u = lr+mt.$$

For this value of γ, q, p and ϵ are given by

$$2\pi q = lmf''/f$$

$$8\pi p = (l^2-m^2)f''/f$$

$$8\pi\epsilon = -2l^2\frac{f''}{f} - 2lm\frac{f''}{f} + (m^2-l^2)\frac{f'^2}{f^2} + \frac{1}{f^2}.$$

If $p = 0$, which implies a distribution of discrete particles, we have from (4.6c)

$$\gamma_{44} - \gamma_{11} = 0$$

for which the most general solution is

$$\gamma = f(r+t) + g(r-t).$$

Equation (4.7) reduces to

$$4f''(f+g) - 4f'\dot{g} + 1 = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (4.9)$$

where

$$f' \equiv \frac{df}{d(r+t)}, \quad \dot{g} \equiv \frac{dg}{d(r+t)}.$$

Differentiating (4.9) with respect to $(r-t)$, we get

$$f''\dot{g} - f'\ddot{g} = 0$$

or

$$\frac{f''}{f'} = \frac{\ddot{g}}{\dot{g}} = k \text{ (constant)}.$$

Then

$$f = \frac{e^{\{k(r+t)+k_1\}}}{k} + k_2, \quad g = \frac{e^{\{k(r-t)+k_3\}}}{k} + k_4.$$

Equation (4.9) becomes

$$4e^{2\{k(r+t)+k_1\}} + 4(k_2+k_4)e^{\{k(r+t)+k_1\}} + 1 = 0$$

which implies $\exp\{k(r+t)+k_1\} = \text{constant}$.

The above equation on further examination leads to the conclusion that discrete particle distribution with geodetic congruence associated with matter is not consistent with spherical gravitational collapse with escaping neutrinos.

In particular, when $g = 0$, ϵ and q can be calculated directly from (4.6a, d). In this case it is found that q turns out to be negative.

Case (*vi*)—$\sigma_{11} = 0$.

Equation (3.13) gives

$$\beta_4 = \frac{2\gamma_4}{\gamma}$$

which on integration gives

$$e^{\beta/2} = g(r)\gamma(r, t).$$

Now (3.1) becomes

$$ds^2 = -e^{2\alpha}dt^2 + \gamma^2\{g^2\,dr^2 + d\theta^2 + \sin^2\theta\,d\phi^2\}$$

which can be written as

$$ds^2 = -e^{2\alpha}\,dt^2+\gamma^2\{dr^2+d\theta^2+\sin^2\theta\,d\phi^2\}. \qquad .. \qquad .. \quad (4.10)$$

For this metric we have

$$u^1 = u^2 = u^3 = 0, \quad u^4 = e^{-\alpha}$$

$$l_1 = l_4 = \bar{l}_1 = \bar{l}_4 = k_2 = k_3 = m_2 = m_3 = 0$$

$$l_2 = \frac{l_3}{i\sin\theta} = \bar{l}_2 = -\frac{\bar{l}_3}{i\sin\theta} = k_1 = m_1 = \frac{\gamma}{\sqrt{2}}$$

$$k_4 = -m_4 = -\frac{e^{\alpha}}{\sqrt{2}}$$

$$n = \frac{F(r)}{\gamma^3}$$

$$c = \frac{e^{-\alpha}}{8\pi F}\left\{\gamma_4\left(-1+4e^{-2\alpha}\gamma\gamma_{44}-6e^{-2\alpha}\alpha_4\gamma\gamma_4+3e^{-\alpha}\gamma_{14}-2e^{-3\alpha}\gamma\gamma_4^2\right.\right.$$
$$\left.+2e^{-2\alpha}\gamma_4^2-\frac{3\gamma_{11}}{\gamma}+\frac{4\gamma_1^2}{\gamma^2}-2e^{-2\alpha}\gamma_1\gamma_4+2e^{-2\alpha}\gamma\gamma_{14}\right)+2\gamma_{114}$$
$$\left.-\frac{2\gamma_1\gamma_{14}}{\gamma}+2e^{-\alpha}\gamma\gamma_{144}-2e^{-\alpha}\gamma_1\gamma_{44}\right\}$$

$$-8\pi\frac{q}{2} = \frac{2e^{-\alpha}}{\gamma}\left(-\frac{\gamma_{14}}{\gamma}+\frac{\gamma_1\gamma_4}{\gamma^2}-e^{-\alpha}\frac{\gamma_4^2}{\gamma}\right)$$

$$-8\pi\left(p+\frac{q}{2}\right) = e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma}+\frac{\gamma_4^2}{\gamma^2}-\frac{2\gamma_4}{\gamma}\alpha_4\right)-\frac{1}{\gamma^2}\left(\frac{\gamma_1^2}{\gamma^2}-2\frac{\gamma_1\gamma_4}{\gamma}e^{-\alpha}\right)+\frac{1}{\gamma^2}$$

$$-8\pi p = e^{-2\alpha}\left(\frac{2\gamma_{44}}{\gamma}+\frac{\gamma_4^2}{\gamma^2}-\frac{2\gamma_4}{\gamma}\alpha_4\right)-\frac{1}{\gamma^2}\left(-\frac{\gamma_{11}}{\gamma}+\frac{\gamma_1^2}{\gamma^2}+e^{-\alpha}\gamma_{14}\right)$$

$$8\pi\left(\epsilon+\frac{q}{2}\right) = 3e^{-2\alpha}\frac{\gamma_4^2}{\gamma^2}+\frac{1}{\gamma^2}\left(-\frac{2\gamma_{11}}{\gamma}+\frac{\gamma_1^2}{\gamma^2}\right)+\frac{1}{\gamma^2}$$

$$v_I = 3e^{-\alpha}\gamma_4/\gamma$$

$$v_{II} = \frac{1}{\sqrt{2}}\left(\frac{\gamma_1}{\gamma^2}+e^{-\alpha}\frac{\gamma_4}{\gamma}\right)$$

and (3.11) and (4.1) become

$$\gamma_4+e^{\alpha}\alpha_1 = 0 \qquad .. \qquad .. \qquad .. \qquad .. \quad (4.11)$$

$$e^{-\alpha}\gamma_{14}+2e^{-2\alpha}\gamma_4^2+\frac{\gamma_{11}}{\gamma}-\frac{2\gamma_1^2}{\gamma^2}+1 = 0. \qquad .. \qquad .. \quad (4.12)$$

One solution of (4.11) is

$$\gamma = -F'f, \quad e^{\alpha} = F\dot{f}$$

where

$$F \equiv F(r), \quad f \equiv f(t), \quad F' \equiv \frac{dF}{dr}, \quad \dot{f} \equiv \frac{df}{dt}.$$

Equation (4.12) reduces to

$$\frac{F'''}{F'} - \frac{2F''^2}{F'^2} - \frac{F''}{F} + \frac{2F'^2}{F^2} + 1 = 0$$

and

$$2\pi q = \frac{1}{F^2 f^2}$$

$$8\pi p = -\frac{1}{F^2 f^2} + \frac{1}{F'^2 f^2}\left(\frac{F'''}{F'} - \frac{F''^2}{F'^2}\right) + \frac{F''}{f^2 F F'^2}$$

$$8\pi \epsilon = \frac{1}{F^2 f^2} + \frac{1}{F'^2 f^2}\left(-\frac{2F'''}{F'} + \frac{F''^2}{F'^2}\right) + \frac{1}{f^2 F'^2}$$

which demonstrates the possibility of neutrino emission in this case.

References

Chiu, H. Y. (1964). Supernovae, neutrinos and neutron stars. *Ann. Phys.*, **26**, 364.

Misner, C. W. (1965). Relativistic equations for spherical gravitational collapse with escaping neutrinos. *Phys. Rev.*, **137**, B 1360.

Vaidya, P. C. (1953). 'Newtonian' time in general relativity. *Nature, Lond.*, **171**, 260.

Witten, L. (1962). Introduction to Current Research. John Wiley & Sons Inc., New York, pp. 57, 58.

COUPLED VIBRATIONS OF A SLENDER BEAM IN A CENTRIFUGAL FORCE FIELD

by J. S. Tomar, *Department of Mathematics, University of Roorkee, Roorkee*

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The differential equations for the coupled bending and torsional vibrations of a slender beam in a centrifugal force field are obtained. A method based on the Rayleigh's quotient is used to obtain an upper bound for the fundamental frequency, and the accuracy of this upper bound is investigated by considering the application of the method to uncoupled vibration.

Notation

x = radial co-ordinate
Y = tangential co-ordinate
z = axial co-ordinate
v = deflection of elastic axis in Y direction
e = axis of minor moment of inertia I min
v' = deflection perpendicular to axis e
l = length of the blade in radial direction
m = mass per unit length of the beam
E = modulus of elasticity
I = moment of inertia of the cross-section about the axis perpendicular to the plane of bending
G = shear modulus of rigidity
J = a constant depending upon the cross-section of the beam such that GJ is the torsional rigidity
δ = distance between the elastic axis and the centroidal axis
θ = rotation of the beam cross-section about the elastic axis
S = area of cross-section of the beam
I_θ = mass moment of inertia about the elastic axis per unit length
t = time
M = moment of centrifugal force at the point x
ξ = dimensionless variable $\xi = x/l$
ω = frequency of vibration

Introduction

The analysis presented in this paper considers vibration of a slender beam that could represent a turbine blade of simple geometry. The beam is attached

to a disc of radius r_0 as indicated in Fig. 1 and the disc rotates with angular velocity Ω. The shear centre of each cross-section does not coincide with the centre of gravity, consequently the torsional and bending oscillations are 'coupled'.

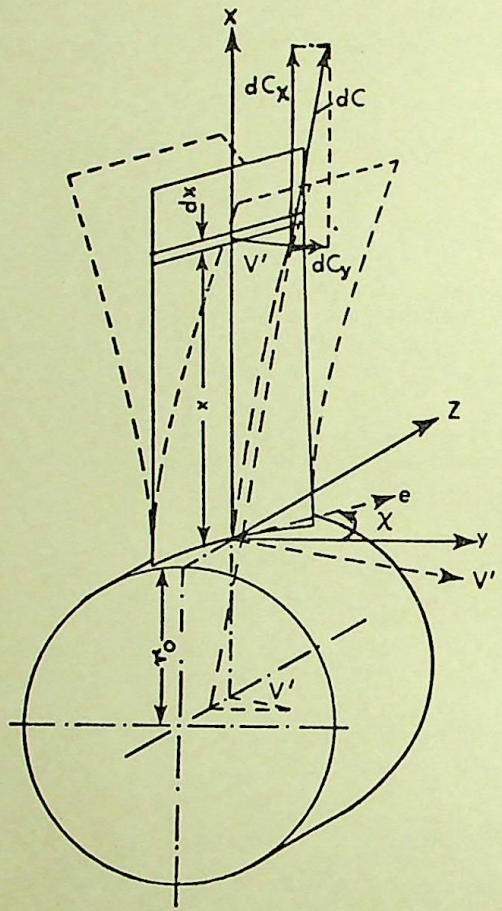

FIG. 1. Cantilever beam attached to a circular disc.

For the analysis it is assumed that S, I and J are constants. This, however, is not an essential assumption. If S, I and J are functions of x, the amount of computing work is greater than for S, I and J independent of x.

THE DIFFERENTIAL EQUATIONS

The differential equation for the deflected form of the neutral axis of a bar according to the elementary theory of bending is

$$EI\frac{\partial^4 v}{\partial x^4} = w \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where w is the intensity of the distributed load.

If the load is distributed along the centroidal axis, the given load can be replaced by the same load distributed along the shear-centre axis, and a torque of intensity $w \cdot \delta$ distributed along the same axis.

Let the x-axis coincide with the shear-centre axis. Since the torsion is not uniform, the relation between the variable torque T and the angle of twist θ is given by (Timoshenko 1955)

$$T = GJ\frac{\partial\theta}{\partial x} - C_1\frac{\partial^3\theta}{\partial x^3}$$

where GJ is the torsional rigidity for uniform torsion and C_1 is the warping rigidity. Differentiation of this equation with respect to x gives

$$GJ\frac{\partial^2\theta}{\partial x^2} - C_1\frac{\partial^4\theta}{\theta x^4} = w \cdot \delta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

with the positive torque taken as shown in Fig. 2.

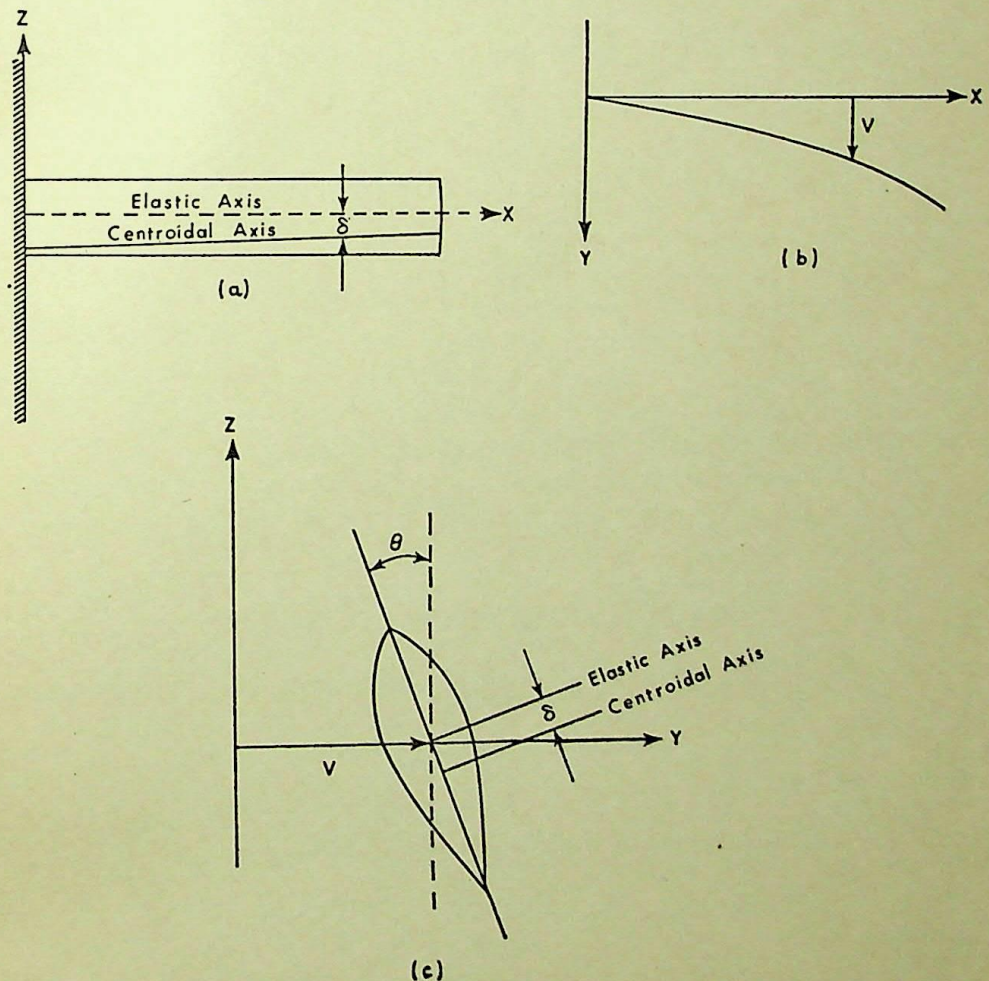

FIG. 2. Deflection and rotation of a cantilever beam from and about elastic axis.

For a vibrating bar the intensity of the inertia force is

$$-m\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)$$

and the intensity of the inertia moment about the x-axis is

$$-I_\theta\frac{\partial^2\theta}{\partial t^2}\,.$$

The following differential equations for the coupled bending and torsional vibrations are obtained by replacing the statical load in eqns. (1) and (2) by the inertia forces,

$$EI\frac{\partial^4 v}{\partial x^4}=-m\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta) \qquad .. \quad .. \quad .. \quad (3a)$$

$$GJ\frac{\partial^2\theta}{\partial x^2}-C_1\frac{\partial^4\theta}{\partial x^4}=m\,.\,\delta\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)+I_\theta\frac{\partial^2\theta}{\partial t^2} \qquad .. \quad .. \quad (3b)$$

Referring to Fig. 1 the centrifugal force dc exerted by a mass element of the radial length $d\eta$ at $x=\eta$ is

$$dc=m\Omega^2\,(r_0+\eta)\,d\eta$$

If the deflection of the blade at time t is $v'(x,\,t)$, the tangential and radial components of dc are

$$dc_Y=dc\frac{v'\sin\chi}{r_0+\eta}$$

and

$$dc_x=dc\left[1-\left(\frac{v'\sin\chi}{r_0+\eta}\right)^2\right]^{\frac{1}{2}}$$

respectively, where χ is the angle between Y and e axes.

If it is assumed that $\sin\chi\ll 1$, $dc_x=dc$. The components c_Y and c_x applied at x follow by integrating from $\eta=x$ to $\eta=l$ and are

$$c_x=m\Omega^2[r_0(l-x)+\tfrac{1}{2}(l^2-x^2)] \qquad .. \quad .. \quad .. \quad (4)$$

$$c_Y=m\Omega^2\sin\chi\int_x^l v'(x,\,t)\,d\eta \qquad .. \quad .. \quad .. \quad (5)$$

Referring to Fig. 3 the moment dM about the minor principal axis of inertia is

$$dM=c_x dv'(x,\,t)-c_Y\sin\chi\,dx \qquad .. \quad .. \quad .. \quad (6)$$

Eqns. (4)–(6) give

$$\frac{\partial^2 M}{\partial x^2}=m\Omega^2\left[\left\{r_0(l-x)+\tfrac{1}{2}(l^2-x^2)\frac{\partial^2 v'}{\partial x^2}-(r_0+x)\frac{\partial v'}{\partial x}+\sin^2\chi v'\right\}\right]$$

If χ is a small angle and $v'=v$ this may be written as

$$\frac{\partial^2 M}{\partial x^2}=m\Omega^2\left[r_0(l-x)+\tfrac{1}{2}(l^2-x^2)\frac{\partial^2 v}{\partial x^2}-(r_0+x)\frac{\partial v}{\partial x}\right] \qquad .. \quad .. \quad (7)$$

The second derivative of the moment M with respect to x can be considered as a lateral load depending upon the centrifugal effect.

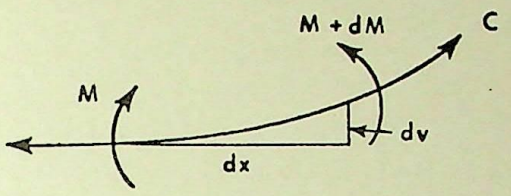

FIG. 3. Showing moment about the minor principal axis of inertia.

The R.H.S. of eqn. (3a) represents an inertia loading, and eqn. (3a) must be modified by adding the lateral load, $\frac{\partial^2 M}{\partial x^2}$, if the centrifugal force effect is to be considered and the governing equations then become

$$\left.\begin{aligned} EI\frac{\partial^4 v}{\partial x^4} &= -m\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)+\frac{\partial^2 M}{\partial x^2} \\ GJ\frac{\partial^2\theta}{\partial x^2}-C_1\frac{\partial^4\theta}{\partial x^4} &= m\,.\,\delta\frac{\partial^2}{\partial t^2}(v+\delta\,.\,\theta)+I_\theta\frac{\partial^2\theta}{\partial t^2} \end{aligned}\right\} \quad .. \quad .. \quad (8)$$

where $\frac{\partial^2 M}{\partial x^2}$ is given by eqn. (7).

DETERMINATION OF NATURAL FREQUENCIES

The equations are now put in terms of a dimensionless variable $\xi = x/l$ and the substitutions,

$$\beta^2 = \frac{EI}{ml^4},\ \gamma^2 = \frac{GJ}{ml^2},\ C_2 = \frac{C_1}{ml^4},\ \text{and}\ I_\theta' = \frac{I_\theta}{m}.$$

are used.

Eqns. (8) become

$$\left.\begin{aligned} &\beta^2\frac{\partial^4 v}{\partial\xi^4}-\Omega^2\left[\left(\frac{r_0}{l}+\frac{1}{2}-\frac{r_0}{l}\xi-\frac{1}{2}\xi^2\right)\frac{\partial^2 v}{\partial\xi^2}-\left(\frac{r_0}{l}+\xi\right)\frac{\partial v}{\partial\xi}\right] \\ &\qquad +\frac{\partial^2 v}{\partial t^2}+\delta\frac{\partial^2\theta}{\partial t^2}=0 \\ &\gamma^2\frac{\partial^2\theta}{\partial\xi^2}-C_2\frac{\partial^4\theta}{\partial\xi^4}-\delta\frac{\partial^2 v}{\partial t^2}-(\delta^2+I_\theta')\frac{\partial^2\theta}{\partial t^2}=0 \end{aligned}\right\} \quad .. \quad (9)$$

The solutions of eqns. (9) are of the form:

$$\left.\begin{aligned} v(\xi,t) &= Af(\xi)e^{i\omega t} \\ \theta(\xi,t) &= B\phi(\xi)e^{i\omega t} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (10)$$

where A and B are constants which are not independent; and $f(\xi)$ and $\phi(\xi)$ are functions of ξ only.

The functions $f(\xi)$ and $\phi(\xi)$ satisfy all the boundary conditions of the beam which are as follows:

$$\left.\begin{aligned} v(0) = \frac{\partial v}{\partial \xi}(0) = \theta(0) = \frac{\partial^2 \theta}{\partial \xi^2}(0) = 0 \\ \frac{\partial^2 v}{\partial \xi^2}(1) = \frac{\partial^3 v}{\partial \xi^3}(1) = \frac{\partial \theta}{\partial \xi}(1) = \frac{\partial^3 \theta}{\partial \xi^3}(1) = 0 \end{aligned}\right\} \quad .. \quad .. \quad (11)$$

For an approximate determination of the fundamental frequency $f(\xi)$ is chosen as the shape function for the fundamental mode of uncoupled bending vibration and $\phi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration of a uniform cantilever beam. These shape functions satisfy the boundary conditions (11) and are

$$\left.\begin{aligned} f(\xi) &= \cosh \lambda\xi - \cos \lambda\xi - 0{\cdot}73410(\sinh \lambda\xi - \sin \lambda\xi) \\ \text{and} \\ \phi(\xi) &= \sin \frac{\pi}{2}\xi \end{aligned}\right\} \quad .. \quad .. \quad (12)$$

where $\lambda = 1{\cdot}87510$.

Substitution of eqns. (10) in eqns. (9) gives:

$$\left.\begin{aligned} &\left[\beta^2 \frac{d^4 f}{d\xi^4} - \Omega^2 \left(\frac{r_0}{l} + \frac{1}{2} - \frac{r_0}{l}\xi - \frac{1}{2}\xi^2\right)\frac{d^2 f}{d\xi^2} + \Omega^2 \left(\frac{r_0}{l} + \xi\right)\frac{df}{d\xi} - \omega^2 f\right] A - \delta\omega^2 \phi B = 0 \\ &\delta\omega^2 f.A + \left[\gamma^2 \frac{d^2\phi}{d\xi^2} - C_2 \frac{d^4\phi}{d\xi^4} + (\delta^2 + I_\theta')\omega^2 \phi\right] B = 0 \end{aligned}\right\} \quad (13)$$

Equations (13) can be solved for ω^2 but the result is a function of ξ, since f and ϕ are not the exact shape function. This difficulty can be overcome (Fung 1955) by multiplying the first and the second of eqns. (13) by f and ϕ respectively, and integrating with respect to ξ from 0 to 1. The method results in the familiar Rayleigh quotient (Collatz 1960) when applied to uncoupled problems, and is an extension of Rayleigh's method to the coupled problem. The following equations are obtained:

$$\left.\begin{aligned} (a_1 - a_2 + a_3 - a_4\omega^2)A - a_5\omega^2 B = 0 \\ -a_6\omega^2 A + (a_7 + a_8 - a_9\omega^2)B = 0 \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (14)$$

where

$$a_1 = \beta^2 \int_0^1 \frac{d^4 f}{d\xi^4} f.d\xi = \beta^2 \int_0^1 \left(\frac{d^2 f}{d\xi^2}\right) d\xi$$

$$a_2 = \Omega^2 \int_0^1 \left(\frac{r_0}{l} + \frac{1}{2} - \frac{r_0}{l}\xi - \frac{1}{2}\xi^2\right)\frac{d^2 f}{d\xi^2} . f . d\xi$$

$$a_3 = \Omega^2 \int_0^1 \left(\frac{r_0}{l} + \xi\right)\frac{df}{d\xi} . f . d\xi$$

$$a_4 = \int_0^1 f^2 d\xi$$

$$a_5 = a_6 = \delta \int_0^1 \phi \,.\, f \,.\, d\xi$$

$$a_7 = -\gamma^2 \int_0^1 \frac{d^2\phi}{d\xi^2} \,.\, \phi \,.\, d\xi = \gamma^2 \int_0^1 \left(\frac{d\phi}{d\xi}\right)^2 d\xi$$

$$a_8 = C_2 \int_0^1 \frac{d^4\phi}{d\xi^4} \,.\, \phi \,.\, d\xi$$

$$a_9 = (\delta^2 + I_\theta') \int_0^1 \phi^2 d\xi$$

For a nontrivial solution A and B must not both vanish, consequently the determinant of the coefficients of eqns. (14) must be zero:

$$\begin{vmatrix} a_1 - a_2 + a_3 - a_4\omega^2 & -a_5\omega^2 \\ -a_6\omega^2 & a_7 + a_8 - a_9\omega^2 \end{vmatrix} = 0$$

i.e.
$$P\omega^4 - Q\omega^2 + R = 0 \qquad .. \qquad .. \qquad .. \qquad (15)$$

where

$$P = a_4a_9 - a_5a_6, \quad Q = (a_1 - a_2 + a_3)a_9 + (a_7 + a_8)a_4$$

and

$$R = (a_1 - a_2 + a_3)(a_7 + a_8)$$

The solution of eqn. (15) is

$$\omega^2 = \frac{Q \pm \sqrt{(Q^2 - 4PR)}}{2P} \qquad .. \qquad .. \qquad .. \qquad (16)$$

The right-hand side of eqn. (16) is positive since it may be shown that $Q^2 - 4PR > 0$.

The smaller of the two values of ω^2 given by eqn. (16) is an upper bound for the frequency of the fundamental mode of vibration. The larger of the two ω^2 values is an upper bound for the next higher mode of vibration.

Also, for the larger value of ω^2, from eqns. (14), A and B have opposite signs and for the smaller value of ω^2 they are of the same sign. The corresponding two configurations are shown in Fig. 4.

Numerical Example

A numerical example for the coupled torsional vibrations of a slender rotating beam is now presented. The frequencies are computed from eqn. (16), and the cross-section of the blade is taken as a semi-circle of radius a_0 and thickness t_1. The physical constants of the blade except the additional assumed value of δ are taken from Grammel (Biezeno and Grammel 1954).

$l = 4{\cdot}41$ in $\qquad E = 29{\cdot}20 \times 10^6$ lb/in^2

$S = 0{\cdot}1488$ in^2 $\qquad I = 0{\cdot}001845$ in^4

$a_0 = 0{\cdot}23885$ in $\qquad G = 11{\cdot}53 \times 10^6$ lb/in^2

$t_1 = 0{\cdot}19706$ in $\qquad J = 0{\cdot}00193$ in$^4 = \dfrac{At_1^2}{3}$

$m = 0{\cdot}00011$ lb $\qquad \delta = 0{\cdot}15$ in

$r_0 = 29{\cdot}10$ in $\qquad = 314$ sec^{-1}

$$I_\theta = 0{\cdot}14834\, m \text{ lb/in}^2 = m\left\{a_0^2 + \left(\frac{2a_0}{\pi} + 0{\cdot}15\right)^2\right\}$$

$$C_1 = 0{\cdot}00014 \times 10^6 \text{ lb/in}^4 = Ea_0{}^5 t_1 \left(\frac{\pi^3}{12} - \frac{8}{\pi}\right)$$

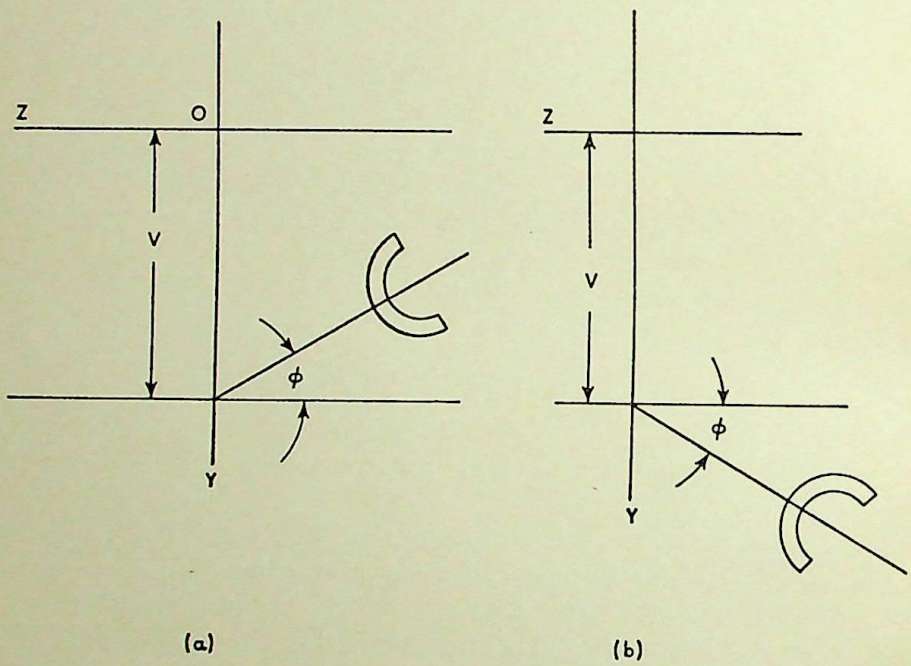

FIG. 4. Configuration corresponding to two values of ω^2.

With these values, the various parameters used in the frequency equation are as follows:

$a_1 = 16{\cdot}20711 \times 10^6$ $\qquad a_5 = a_6 = 0{\cdot}10151$

$a_2 = 0{\cdot}30724 \times 10^6$ $\qquad a_7 = 12{\cdot}75067 \times 10^6$

$a_3 = 1{\cdot}44909 \times 10^6$ $\qquad a_8 = 0{\cdot}01021 \times 10^6$

$a_4 = 1$ $\qquad a_9 = 0{\cdot}08542$

and from eqn. (16)

$$\omega_1^2 = 1{\cdot}70830 \times 10^7 \text{ sec}^{-2}$$

and

$$\omega_2^2 = 1{\cdot}72511 \times 10^8 \text{ sec}^{-2}$$

which are the upper bounds of the frequencies corresponding to the first two modes of the torsional coupled vibrations of the rotating blade.

Discussion and Concluding Remarks

For the uncoupled case (i.e. $\delta = 0$), eqns. (13) are two separate eigenvalue problems that satisfy the conditions of self-adjointness and full definiteness (Collatz 1960). The first of eqns. (13) when $\delta = 0$ is

$$\beta^2 \frac{d^4 f}{d\xi^4} - \Omega^2 \left\{ \left(\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\,\xi - \tfrac{1}{2}\xi^2 \right) \frac{d^2 f}{d\xi^2} - \left(\frac{r_0}{l} + \xi \right) \frac{df}{d\xi} \right\} = \omega^2 f$$

with boundary conditions $f(0) = f'(0) = f''(1) = f'''(1) = 0$
This can be written as

$$U[f] = \omega^2 V[f]$$

where U and V are differential operators and the boundary conditions as

$$B[f] = 0.$$

A sequence of functions $f_1, f_2, f_3 \ldots$ can be obtained from an arbitrary f_0 and application of the boundary value problems

$$\left.\begin{aligned} U[f_k] &= V[f_{k-1}] \\ B[f_k] &= 0. \end{aligned}\right\} \quad k = 1, 2, 3 \ldots$$

If the ratio f_{k-1}/f_k tends to a constant as k increases, the sequence converges, and the function f_k converges to the form of the first eigen-function f. The quotient $U[f_k]/V[f_k]$ should provide an approximation to ω_s^2, where ω_s is first eigenvalue, but as it is still a function of ξ the numerator and denominator are each multiplied by f_k and integrated with respect to ξ from 0 to 1 to give Rayleigh's quotient which is an upper bound for ω_s^2, that is

$$A = \frac{\int_0^1 f_k U[f_k] d\xi}{\int_0^1 f_k V[f_k] d\xi} = \frac{\int_0^1 f_k V[f_{k-1}] d\xi}{\int_0^1 f_k V[f_k] d\xi}$$

The form of the integrals appearing in the above expressions for A may be denoted by

$$\left.\begin{aligned} b_{2k-1} &= \int_0^1 f_k V[f_{k-1}] d\xi \\ b_{2k} &= \int_0^1 f_k V[f_k] d\xi \\ b_{2k+1} &= \int_0^1 f_{k+1} V[f_k] d\xi \end{aligned}\right\} (k = 0, 1, 2, \ldots)$$

and are called Schwarz constants (Collatz 1960) and Λ is the quotient of two successive Schwarz constants

$$\Lambda = \mu_{2k} = \frac{b_{2k-1}}{b_{2k}}$$

The quotients $\mu_{2k-1} = b_{2k-2}/b_{2k-1}$, $\mu_{2k} = b_{2k-1}/b_{2k}$, $\mu_{2k+1} = b_{2k}/b_{2k+1}$, ... are known as Schwarz's quotients (Collatz 1960) and the even numbered quotients are identical with Rayleigh's quotients.

Bounds for the first eigenvalue ω_s^2 can be obtained from the quotients μ_k, μ_{k+1} and l_2 where l_2 is a lower bound for the second eigenvalue and such that $l_2 > \mu_{k+1}$. The bounds are given by (Collatz 1960)

$$\mu_{k+1} - \frac{\mu_k - \mu_{k+1}}{\frac{l_2}{\mu_{k+1}} - 1} \leq \omega_s^2 \leq \mu_{k+1} \; (k = 1, 2, 3 \ldots)$$

For numerical computation of the quotients f_1 is taken as the shape function for free vibration of a uniform cantilever, that is

$$f_1 = \cosh 1{\cdot}8751\xi - \cos 1{\cdot}8751\xi - 0{\cdot}7341 \, (\sinh 1{\cdot}8751\xi - \sin 1{\cdot}8751\xi)$$

and is a non-zero function and satisfies all boundary conditions and possesses continuous derivatives.

Also, a function f_0 such that

$$U[f_1] = V[f_0]$$

or

$$\beta^2 \frac{d^4 f_1}{d\xi^4} - \Omega^2 \left\{ \left(\frac{r_0}{l} + \tfrac{1}{2} - \frac{r_0}{l}\xi - \tfrac{1}{2}\xi^2 \right) \frac{d^2 f_1}{d\xi^2} - \left(\frac{r_0}{l} + \xi \right) \frac{df_1}{d\xi} \right\} = f_0$$

can be readily obtained.

In this special eigenvalue problem with $V[f]$ the condition

$$\int_0^1 (f_0 V[f_1] - f_1 V[f_0]) \, d\xi = 0$$

is satisfied and according to Collatz (1960) f_0 need not satisfy any boundary conditions.

Using the same numerical values for the physical constants of the beam already considered but with $\delta = 0$ Schwarz constants and quotients are

$$b_0 = \int_0^1 f_0^2 \, d\xi = 3{\cdot}02437 \times 10^4 \qquad \mu_1 = \frac{b_0}{b_1} = 1{\cdot}74325 \times 10^7$$

$$b_1 = \int_0^1 f_0 f_1 \, d\xi = 1{\cdot}73490 \times 10^7 \qquad \mu_2 = \frac{b_1}{b_2} = 1{\cdot}73490 \times 10^7$$

$$b_2 = \int_0^1 f_1^2 \, d\xi = 1$$

and an upper bound is

$$\mu_1 \geq \mu_2 \geq \omega^2 \text{ i.e. } 1{\cdot}74325 \times 10^7 \geq 1{\cdot}73490 \times 10^7 \geq \omega^2$$

For calculating lower bound from the expression

$\mu_2 - \frac{\mu_1 - \mu_2}{l_2/\mu_2 - 1} \leq \omega_s^2$, it is not essential to have a close lower bound l_2 for the second eigenvalue and even a rough value for l_2 is justified since changes in l_2 have little effect on the lower bound calculated from the above expression when l_2 is appreciably greater than μ_2.

Further, it is quite reasonable to assume that for flexural vibrations, the second eigenvalue for free vibration of the uniform beam is a good enough approximation for the lower bound of the second eigenvalue of the rotating cantilever beam. Hence, the calculated value for l_2 is $6{\cdot}36515 \times 10^8$ sec^{-2}.

Thus, the frequency for fundamental mode of uncoupled vibration is bounded as

$$1{\cdot}73467 \times 10^7 \text{ sec}^{-2} \leq \omega^2 \leq 1{\cdot}73490 \times 10^7 \text{ sec}^{-2}$$

The above result shows that the method provided very narrow limits for fundamental frequency of uncoupled vibrations in the presence of a centrifugal force field.

The fundamental frequency for coupled vibrations could not be bounded at present, but the above computation indicates that Rayleigh's quotient obtained from the shape function f_1 is a very close upper bound for the first eigenvalue ω_s^2 for the uncoupled problem. Consequently, it seems reasonable to believe that the method results in a close upper bound for the fundamental frequency for the coupled problem, especially if δ is small.

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References

Biezeno, C. B., and Grammel, R. (1954). Engineering Dynamics. Blackie and Sons, London, p. 124.

Collatz, L. (1960). The Numerical Treatment of Differential Equations. Springer-Verlag, Berlin, p. 228.

Fung, Y. C. (1955). An Introduction to the Theory of Aero-elasticity. John Wiley & Sons Inc., New York, p. 52.

Timoshenko, S. (1955). Vibration Problems in Engineering. D. Van Nostrand & Co., New York, p. 407.

PETROGRAPHY OF THE TAL SERIES, SOUTH OF THE VILLAGE KOTIDHAMAN, SIRMUR DISTRICT, HIMACHAL PRADESH, WITH A NOTE ON ITS PROBABLE ENVIRONMENTAL CONDITION OF DEPOSITION

by S. KANJILAL, *Department of Geology, Banaras Hindu University, Varanasi 5*

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Petrography of the Tal Series south of the village Kotidhaman (77° 33′ : 30° 38·5′) has been studied here in detail. On the basis of this study they have been subdivided into three units as against two by Auden (1934). The study also revealed that the rocks originally described as 'graywacke' and 'quartzite' are actually siltstone and a variety of sandstone with a very high detrital quartz percentage. A probable environmental condition of deposition of these rocks has been suggested on the basis of characteristic lithologic association and the various sedimentary structures present in the area.

INTRODUCTION

The area under study forms the southernmost part of the 'Western Basin' of the 'Krol Belt' described by Auden (1934) in his classic memoir, 'Geology of the Krol Belt'. There is a beautiful development of the Tal Series in this area. In the present paper a petrographic study of the Tal Series, as occurring to the south of the village Kotidhaman, Sirmur District, Himachal Pradesh, and lying between longitudes 77° 32′ to 77° 35′ and latitudes 30° 36′ to 30° 39′, has been made. Besides the Tal Series some formations of the Krol Series also occur in the area, but they have not been included in the discussion here.

GEOLOGY OF THE AREA

The topography of the area is mountainous and highly rugged with very prominent relief, characterized by escarpments, high hills dissected by deep valleys and spurs (Fig. 1). Except a few, most of the streams are dry during the summer season. The height of the hills gradually rises toward north. The Tal Series, which is a very thick pile of partly argillaceous and partly arenaceous sediments, occurs throughout the area except in the south-western portion (Fig. 2) where the Krol Series is found outcropping. The rocks comprising the Tal Series are folded into a syncline. The beds are found dipping at 25°–35° due south-west, in the northern side of the area, while on the western to southern sides the dip direction changes from N.N.E. to almost

north and the amount of dip varies from 40° to 50°. The Krol Series, too, correspondingly changes its dip direction from N.E. to north.

FIELD RELATIONSHIPS

The Tal Series of the area consists of a great variety of argillaceous and arenaceous rocks. Often they create illusion with regard to their position owing to repetition of sediments of similar nature. All the rock types pass on or grade into one another and thereby create much difficulty in demarcating sharply the contact planes of the different lithic units. The author has, however, included each transition bed into the younger formation, because the transition beds indicate a beginning of a change in lithology from the older sediments.

FIG. 1. Photo illustrating the intricate system of bifurcating valleys—a characteristic feature of the hills where Upper Tals are exposed. Photo taken from the hill near Kuwal village and looking due south.

SUCCESSION

Auden (1934) has proposed a twofold division of the Tal Series of this locality. The Lower Tal formation of Auden, however, can be further subdivided lithologically in view of the presence of different and distinct rock types. The base of the Lower Tal consists of soft black shales with chert but the upper portion is made up of hard and tough siltstones. Although there is no sharp line of junction between the two, and the shales gradually change to siltstones, there seems no justification in combining them together, particularly when they strikingly differ in their physical properties and composition, which are more elaborately discussed ahead. Therefore, the present author has further subdivided the 'Lower Tal' of Auden

into Lower Tal (in restricted sense) and Middle Tal, leaving Auden's 'Upper Tal' as such.

The whole Tal Series rests somewhat unconformably over the Krol C unit of the Krol Series, as evidenced by the difference of dip direction and the presence of a thin conglomerate bed between the two series. This unconformity, however, is not persistent throughout and is found only in the western side of the present area. The Krol C unit in the area consists of sandstones at the bottom and bluish limestones at the top.

After a careful study the following stratigraphic sequence has been established :

Tal Series (Jurassic)	Upper Tal	Calcareous sandstone. Protoquartzite and orthoquartzites with intercalations of red and green shales. Arkose and thin shale beds. Interbedded siltstone and siliceous bands of varying thickness.
	Middle Tal	Thick series of dark coloured siltstone passing upwards to light coloured siltstone, with ripple marks. Banded, black, carbonaceous siltstone passing upwards to dark coloured siltstone.
	Lower Tal	Massive black, carbonaceous silty shales gradually merging upwards into banded, black, carbonaceous siltstones. Soft, black shales interbedded with very thin and irregular chert bed.

——————————Slight unconformity——————————

Krol Series
(Permo–Carboniferous or Triassic ?)

Structures

The structural features of the area can be divided into two broad heads, namely 'tectonic structures' and 'sedimentary structures'. Among the former are included the fault, folds and joints while in the latter mention may be made of bedding, ripple marks, mud cracks and vermicular tubes which may be worm borings.

The Krol C unit is faulted against the other older formations, i.e. Infra Krol, Krol A and Krol B. The Tals, however, have not yielded to stress due to folding. Very small faults, however, which are of the nature of minor

slips along the bedding as well as joint planes could be seen at places. As mentioned earlier, the Tals are folded into syncline. Joints are highly developed in the Tals, particularly in the Middle Tal formation (Fig. 3). Very good dip and strike joints are developed due to which, at times, big rectangular blocks are found. Often very good slickenside markings can be seen due to slipping along the joint planes in the Upper Tal formation.

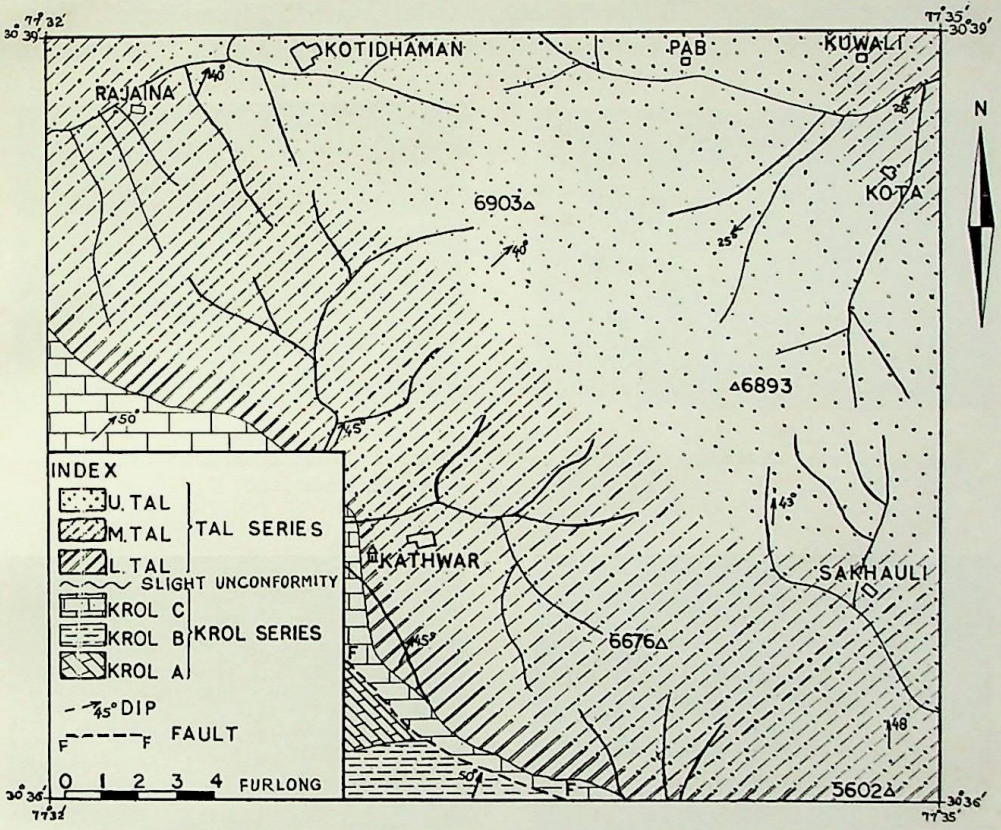

FIG. 2. Geological map of the area south of the village Kotidhaman, Sirmur District, Himachal Pradesh (a part of survey Topo sheet No. 53 F/10).

As regards the sedimentary structures, nature of the bedding is the most striking feature. The most common type is the massive bedding ranging from 0·5 to 1·5 m in thickness, displayed by the middle and upper divisions of Tal Series. Cross-stratification or cross-bedding is frequently seen in the Upper Tal formation (Fig. 4). These cross-beddings consist of 'sets' of beds that are similarly shaped and approximately parallel. The sets are of limited and generally moderate thickness (0·1 to 0·5 m) and are separated from other sets by stratigraphic planes which are, however, comparatively much thinner than the cross-bedded units. The cross-beds are of tabular type (Potter and Pettijohn 1963) with the individual foresets having dips varying from 20° to

25° with respect to the true bedding plane. Their dip direction confirms with that of the bedding plane. The foresets have planer basal to slightly tangential contact with the underlying stratigraphic plane and are truncated upwards against the overlying sets. This feature clearly indicates that the Upper Tal sediments are in a normal sequence of deposition and not inverted. Among some other less frequent sedimentary structures are the ripple marks in the uppermost beds of the Middle Tal and also in Upper Tal formations. These are asymmetrical current ripples with broadly rounded crests. Their certain

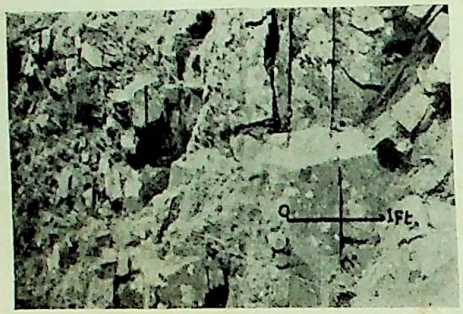

FIG. 3. Photo illustrating jointing in Middle Tal rocks.

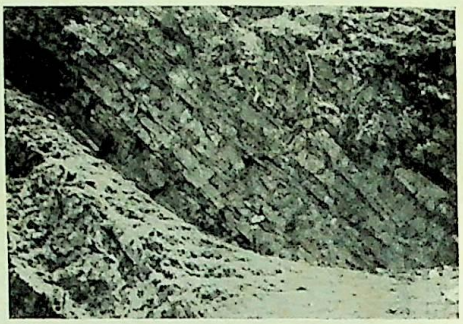

FIG. 4. Beautiful large scale cross-bedded Upper Tal sandstones.

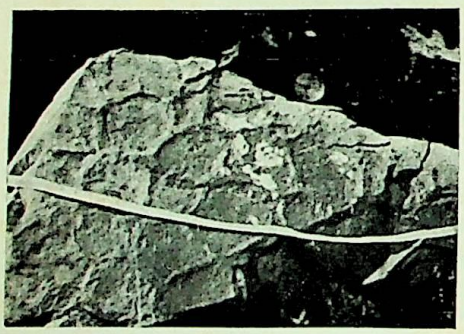

FIG. 5. Photo illustrating mud cracks in red, sandy shale beds of Upper Tal formation.

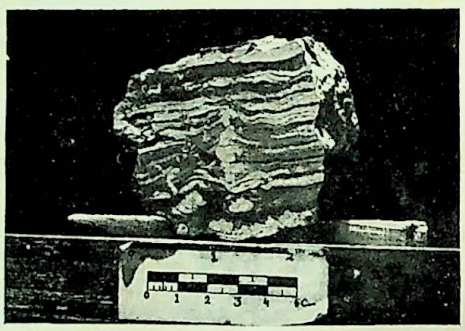

FIG. 6. Polished rock surface showing vermicular tube, nodules and turned up edges of siliceous bands in the red, sandy shale beds.

properties were noticed in the case of Middle Tals only; for example, they have a fairly persistent strike in a direction ENE–WSW, their stoss side direction is NNW, having a very small amplitude as compared to wavelength (ripple index 17·3). Mud cracks are essentially found in the red, sandy shale beds of the Upper Tal formation which indicate that these sediments were exposed to sun (Fig. 5). Peculiar vermicular tubes (which may be worm borings?), whenever seen, are found localized in these red, sandy shale beds (Fig. 6).

TABLE I

Modal analysis of the Tal sediments

Sl. No.	Locality and specimen No.	Quartz %	Felspar %	Mica %	Cement and matrix %	Rock fragment including chert %	Heavy minerals %	Colour and texture	Name
1.	Near the hill 6903 ft.; 57/817	72·88	2·22	1·34	22·42	1·09	0·04	Brownish white; medium grained	Calcareous sandstone
2.	Near the hill 6893 ft.; 57/819	70·34	1·82	1·90	23·70	1·18	0·75	Brown; medium grained	Calcareous sandstone
3.	South-east of Kotidhaman; 57/829	93·70	1·32	0·27	3·66	0·86	0·19	Pale greenish white; medium to coarse grained	Orthoquartzite
4.	Near the hill 6903 ft.; 57/831	93·17	nil	0·51	5·23	0·85	0·24	Dark green; medium grained	Orthoquartzite
5.	South of the hill 6893 ft.; 57/841	89·57	4·38	0·25	2·82	2·85	0·13	Pinkish; very coarse grained	Protoquartzite
6.	North-east of the hill 6893 ft.; 57/843	86·61	1·83	0·68	6·84	3·34	0·70	Green; coarse grained	Protoquartzite
7.	North of Sakhauli; 57/845	91·16	2·23	0·52	3·66	1·76	0·67	Dark olive green; coarse grained	Orthoquartzite
8.	West of Pab; 57/865	82·37	5·90	0·73	5·41	4·57	1·01	Light yellowish brown; medium to coarse grained	Protoquartzite
9.	South-west of Kota; 57/869	90·98	2·78	0·38	3·72	1·44	0·70	Dirty greenish; white; medium grained	Orthoquartzite
10.	S.S.W. of Kota; 57/880	81·59	4·17	0·95	5·61	7·57	0·10	White; medium grained	Protoquartzite
11.	W.S.W. of hill 6903 ft.; 57/821	75·66	18·17	0·26	2·52	3·13	0·26	Pink; very coarse grained	Arkose
12.	West of the hill 6903 ft.; 57/822	77·33	16·49	0·28	2·82	2·60	0·48	Pale pink; coarse grained	Arkose
13.	South of Rajaina; 57/815	60·30	0·40	6·20	32·80	nil	1·2	Gray; fine grained	Siltstone
14.	North-west of Rajaina; 57/859	48·90	nil	8·10	42·90	nil	0·3	Dull, greenish gray; fine grained	Siltstone
15.	South-east of Kuwali; 57/876	50·70	0·20	4·70	43·60	nil	0·7	Dark gray; fine grained	Siltstone

Petrography

For the sake of clarity the petrography of each formation is described below separately:

Lower Tal

The Lower Tal formation consists of chert and black shales. In the lower portion soft black shales are found interbedded with thin irregular chert bands, while in the upper portion massive, black, carbonaceous, silty shales are found. Both the shales and associated cherts are found to be phosphatic. The phosphatic shales are softer and soil the fingers and, therefore, can be easily distinguished from the overlying barren, black shales in the field. The phosphatic cherts occur in two ways; firstly, as nodules which are rather elliptical in shape and range in size from 1 to 3 cm, and secondly, as friable granular variety. The chert beds are very fragmentary and the shales are so soft that neither of them could be sliced properly and examined under microscope.

Overlying the cherts are barren, black, carbonaceous shales. Under microscope nothing more than silt sized quartz grains with some slender flakes of mica in a dark, almost opaque ground of carbonaceous material is visible.

Middle Tal

The black shales, which constitute the uppermost part of the Lower Tal formation, merge very gradually into the banded, black, carbonaceous siltstone, which is a transition zone, and normally it is not easy to separate one from the other. However, considering the lithological characters on the whole, the Middle Tal sediments can be distinguished clearly from those of the Lower Tal. The Middle Tal formation corresponds to the upper part of the Lower Tal of Auden (1934), i.e. graywacke. In the field the mode of occurrence of these rocks can tempt anybody to name them as graywackes. The rocks are hard and massive, virtually structureless, dark gray coloured and break with a sub-conchoidal fracture. All these megascopic characters can be seen in any typical graywacke but the mineralogical composition, as described below, does not fulfil the essential conditions to call it graywacke (Table I). The term graywacke was originally used by Auden with the following explanation: '... No ferromagnesian minerals like augite or hornblende has been found, such as are probably required by the definition of these rocks, but the abundance of chlorite suggests the formation from minerals derived from basic rocks.' But for such assumption evidences such as relics of augite or hornblende in chlorite or pseudomorph of chlorite after the ferromagnesian minerals, etc., are necessary which was never seen in a single instance. Moreover, felspar is rarely seen while rock fragments are totally absent (Kanjilal 1966),

both of which are essential for a rock to be called graywacke (Pettijohn 1957). Due to this mineralogical paradox, size phenomena are used by the present author in giving a new name to these rocks. Since the maximum size of about 80 per cent of the detritus is never more than 0·06 mm, the term 'siltstone' seems to be the most suitable one.

These siltstones are massive, well jointed, dark coloured, micaceous and fine grained. The lower part is black, carbonaceous siltstone marked by fine bandings of lighter shades of gray but of similar composition. They merge upwards into dark coloured siltstones, the upper part of which gradually becomes light coloured and more fine grained. There is a characteristic phenomenon invariably true for the entire Middle Tal formation of the area and that is a decrease in the amount of carbonaceous matter and size of detrital quartz grains, together with an increase in argillaceous matter, when followed upwards. The rocks also respond to dilute HCl, indicating the presence of carbonate.

Under microscope can be seen angular to sub-angular quartz grains, small flakes of micas (generally ferrimuscovite) showing a parallel to sub-parallel orientation which may be due to slight reworking in the basin of deposition (Fig. 7), some chlorite which is undoubtedly detrital and not of secondary

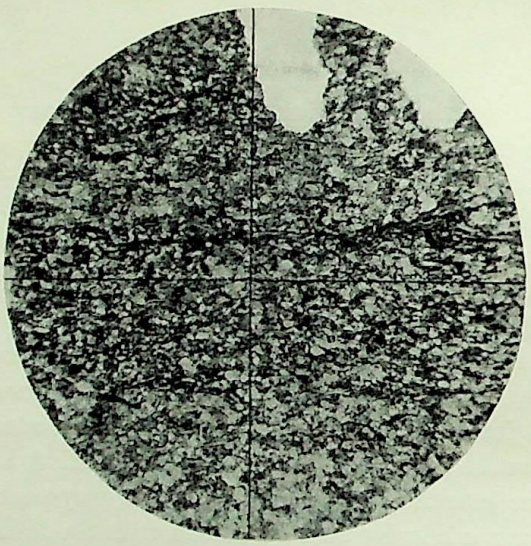

FIG. 7. Dark coloured siltstone, showing some parallel arrangement of micas. Crossed nicols; 38·88×.

origin as advocated by Auden, and a very high percentage of argillaceous matrix. The matrix is a homogeneous mixture of very finely divided pulp of chlorite together with some clay minerals, sericite, mica and extremely fine grained quartz. Carbonate dust is invariably present in the matrix. In the lower horizon the matrix is flooded with carbonaceous material.

Only a few grains of plagioclase felspar are seen scattered in the matrix. Small crystals of pyrite and rarely rounded tourmaline, zircon and iron oxide are seen.

Upper Tal

The Upper Tal formation consists of different types of sandstones. They show a wide range of colours, e.g. white, shades of green, yellow, brown and grey. Pinkish and reddish beds are at places found interbedded with them (Kanjilal 1966). They also have a wide range of size grades, ranging from silt size to pebble size. The upper portion of this division shows some alternate coarse- and fine-grained bands with intercalations of silty, red and green shales.

The microscopic study reveals that the sediments of the Upper Tals should be broadly named as sandstone only. The author wonders if the term 'quartzite', used at several places by Auden for different beds in the Upper Tal formation, stands for a metamorphic rock. (Please see the sequence given by Auden in his memoir, 'Geology of the Krol Belt'.) It is certainly not possible to imagine a metamorphic rock sandwiched between two sedimentary beds and showing a full conformity among them. The rounded grains, overgrowth on the detrital grains and the presence of cement verify that they are still a sedimentary rock and not metamorphosed to quartzite. On the basis of modal analysis, four different rock groups have been made, namely arkose, protoquartzite, orthoquartzite and calcareous sandstone. The terms 'protoquartzite' and 'orthoquartzite' may not be acceptable to all, but the author prefers them as indicating a particular type of sandstone. Krynine (1948) defined orthoquartzite as 'pure sandstone' with 90 per cent or more detrital quartz and protoquartzites as those with 75–90 per cent of quartz and cement below 15 per cent. This mineralogical definition is followed throughout in this paper. Besides those sandstones with more than 12 per cent felspar are considered as arkose (Pettijohn 1954) and those with less than 75 per cent of quartz as only sandstone (e.g. calcareous sandstone).

Roundness of all the above-mentioned four groups has been measured by Powers' method (1953). Sphericity was measured by drawing outlines of grains as seen in thin sections with the help of a camera lucida. The areas of different grains were measured along with the largest diameters of the grains. The sphericity for each grain was calculated by the following formula (Krumbein and Pettijohn 1938):

$$\phi_1 \text{ (sphericity)} = \frac{dc}{Dc}$$

where

dc = diameter of circle of same area as magnified grain projection

$$= \sqrt{4A/\pi}$$

where A is the magnified area of the grain and Dc the diameter of the smallest circle circumscribing grain projection.

Actual roundness and sphericity data for four representative thin sections representing all the types of Upper Tal rocks are given in Tables II and III.

TABLE II

Roundness data

Sl. No.	Specimen No. and name	Numerical mean roundness	Form of grain	Roundness/size relationships
1.	57/819 Calc. sandstone	0·32	Sub-angular	Smaller grains are more angular.
2.	57/831 Orthoquartzite	0·39	Sub-rounded	Both smaller and larger grains are equally rounded.
3.	57/843 Protoquartzite	0·41	Sub-rounded	Roundness varies in the grains of same size.
4.	57/822 Arkose	0·37	Sub-rounded	Both smaller and larger grains are equally rounded.

TABLE III

Sphericity data

Sphericity grade	% of grains in specimen No.			
	57/819	57/831	57/843	57/822
1·0 –0·95	—	7	3	—
0·95–0·90	4	10	5	2
0·90–0·85	6	18	12	6
0·85–0·80	30	25	18	32
0·80–0·75	35	15	26	26
0·75–0·70	20	12	18	14
0·70–0·65	5	8	12	10
0·65–0·60	—	5	6	7
0·60–0·55	—	—	—	3

Petrography of each rock group is dealt below one by one in detail:

(*a*) *Arkose*—After a few tens of metres of interbedding, there is an arkose bed of varying thickness. The rock is very coarse grained, hard and compact and pinkish in colour. The major constituents are quartz and pink felspar. The grains seem to be sub-rounded with naked eye and range in size from 1 to 2·5 mm. Under microscope the rock consists of sub-rounded to rounded quartz and felspar grains. The quartz grains are mostly inclusion

free and have straight to slightly curved contacts. More than 90 per cent of the detrital quartz show undulatory extinction and rest of them are polycrystalline (8 per cent) and non-undulatory type (2 per cent) (Conolly 1965). It is seen that only the highly rounded grains show a non-undulatory extinction. On some detrital grains of quartz, silica overgrowths in the same optical orientation are seen. Two types of felspars are seen, microcline and untwined felspar, whose species could not be determined due to its weathered state. The latter type is much less in amount than the former one. Often rhombic overgrowths are seen on the felspar grains (Fig. 8). Among other

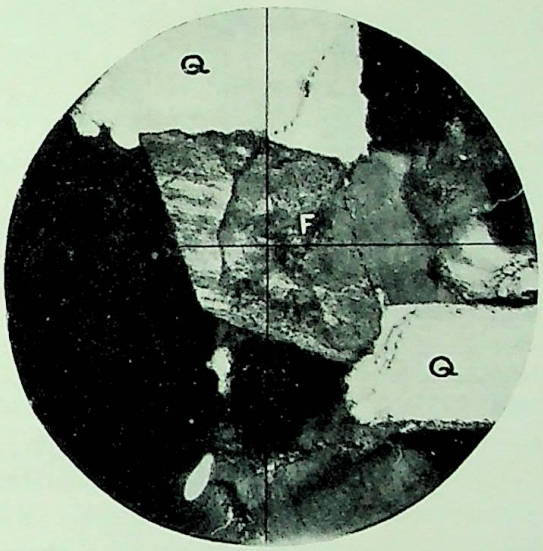

FIG. 8. Photomicrograph showing an authigenic felspar (F) grain at the centre. Two authigenic quartz (Q) grains and cement are also seen. Crossed nicols; 155·55×.

constituents are insignificant amounts of muscovite, granitic rock fragments and cement which is primarily a clay-chlorite mixture.

(*b*) *Protoquartzite and Orthoquartzite*—The protoquartzites and orthoquartzites are of different colour and grain size. Comparatively fine grained rocks are more micaceous. Thin shale partings are always present after about 30 to 40 m of these rocks. In the upper part these rocks contain red coloured sandy shale beds with frequent mud cracks and vermicular tubes. Under microscope this group of rocks consists of medium to coarse grained sediments (0·31 to 2·00 mm) with individual grains having a sub-angular to rounded form. Quartz is the most dominant mineral, most of which is clear and inclusion free. However, few grains show inclusions of zircon, rutile, muscovite, quartz and iron oxide which are either scattered at random or arranged in a linear fashion. The grains show different types of contact (from curved to straight) due to diagenetic effects. Silica overgrowth, which

is always in optical continuity with the nucleus, is very common (Fig. 9). As in arkose, maximum of the detrital quartz grains (90–97 per cent) are of undulatory type. Among rest of them, 0–4 per cent are non-undulatory

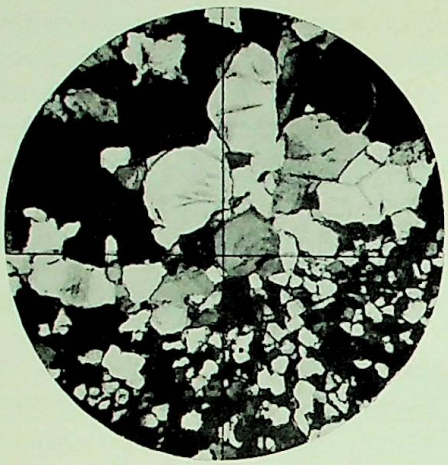

FIG. 9. Orthoquartzite with alternate fine and coarse bands. Also showing rounded grains and secondary overgrowth on quartz. Some quartz grains show a linear inclusion of iron oxide. Crossed nicols; 33·33×.

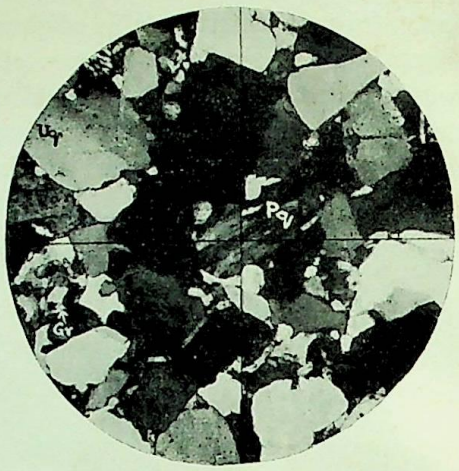

FIG. 10. A typical protoquartzite from the area showing a poly-crystalline quartz (Pq) at the centre, an undulatory quartz (Uq) and a granitic rock-fragment (Gr) in the middle left hand side of the photo. Crossed nicols; 33·33×.

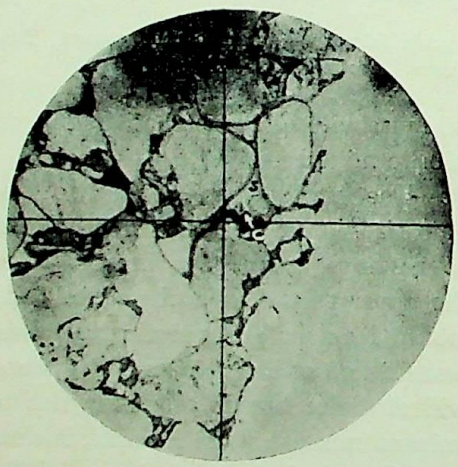

FIG. 11. Secondary silica cement (S) (derived from primary clay-chlorite cement-C) forming a wedge shaped overgrowth over quartz grains. Crossed nicols; 33·33×.

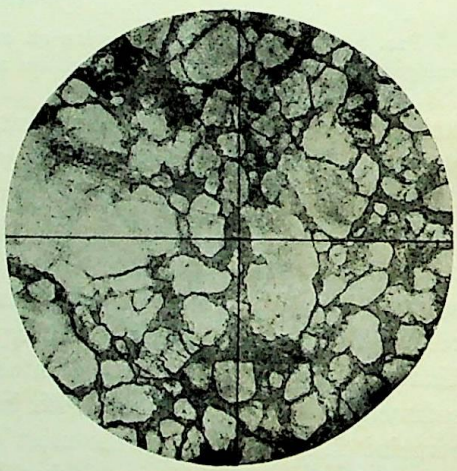

FIG. 12. Another protoquartzite showing clay-chlorite cement. Note how the cement has corroded the quartz grains. Ordinary light; 33·33×.

and 0–10 per cent of polycrystalline types (Fig. 10). The second most prominent mineral is felspar which is chiefly microcline. Orthoclase is altogether absent and plagiclase members are rare. Some microcline grains show inclusions of accicular muscovite. Mica is present in very small amounts except in a few specimens. Muscovite is the chief member which is actually an alteration product of biotite by the removal of iron.

The matrix and cementing material of these rocks are very interesting. They are always less than 15 per cent (Kanjilal 1966). Three types of cement have been noted: (i) primary clay-chlorite cement, (ii) secondary silica cement and (iii) secondary ferruginous cement.

The first group consists of an extremely fine grained mixture of clay and chlorite which appears to be primary and deposited in the basin together with other detritus. This type of cement dominates over the other two types. It is, however, difficult to say that which one of the two (i.e. clay or chlorite) is greater in amount. The silica cement appears here to be secondary and derived from the basin of sedimentation itself by diagenetic changes. Two mechanisms may have operated in the derivation of this cement:

(1) 'Silica leached out from the clay minerals due to diagenetic changes in the basin of deposition.' (Towe 1962). This is evidenced by silica aggregates near or around the clay-chlorite cement (Fig. 11). Where leaching is maximum, relics of clay-chlorite are found within the silica cement.

(2) Silica removed in solution from the point of contacts between two or more than two quartz grains and from the detrital quartz grain itself attacked by the primary cement (Fig. 12).

The ferruginous cement has developed only locally. The probable source may be the iron released from biotite during the process of alteration to ferrimuscovite.

Among other constituents are the rock fragments af schists, shales, granite and chert. These occur as rounded to sub-rounded grains. Several types of heavy minerals including both opaque (like pyrite, iron oxides and ilmenite) and non-opaque (like tourmaline, zircon and rutile, some apatite, garnet and chlorite) varieties are seen.

(c) *Calcareous sandstone*—This is the topmost member of the Upper Tal formation as occurring in the area. It is brownish white in colour, hard and medium grained. The weathered surface is deeply stained with limonitic matter. The term calcareous sandstone is given to it because the quartz content of this rock type is less than 75 per cent and the cement is chiefly calcareous. Under microscope the texture is medium grained, with subangular to sub-rounded grains of quartz and felspar, cemented in a calcareous matrix. The quartz grains are mostly of 0·125 to 0·250 mm size. Besides the calcareous cement which is definitely primary, some limonitic matter also occurs in the matrix.

ENVIRONMENTAL CONDITION OF DEPOSITION

The Lower Tal formation consists of interbedded black shales and cherts which are about 200 ft. thick. This may have been deposited in an euxinic condition (Pettijohn 1957). The association of this black shale with carbonate-sandstone (Krol C) rocks below and the immature rock, siltstone (Auden's greywacke) above it, records a transition between the open sea deposition and craton derived sands and the influx of sediments from a rising island arc. The initial rise of the future arc results in barriers which lead to partial isolation of the locus of deposition and restricted circulation conditions conductive to stagnation of bottom waters and deposition of reduzates and related sediments (James 1954).

When the basin of deposition was almost filled up, a delto-continental condition was probably reached and the mode of deposition of sediments was attended by a change in provenance, during which time the interbedded siltstone and siliceous bands may have been deposited. Interbedded red shale beds of the Upper Tal formation indicate a strongly reducing condition. Mud cracks and the vermicular tubes in them also point to a land or near-land condition under hot and arid climate. Sedimentary parameters of the Upper Tal formation indicate a beach environment.

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REFERENCES

Auden, J. B. (1934). Geology of the Krol Belt. *Rec. Geol. Surv. India*, **67**, 357–454.

Conolly, J. R. (1965). The occurrence of polycrystalline and undulatory extinction in quartz in sandstone. *J. sedim. Petrol.*, **35**, 116–35.

James, H. L. (1954). Sedimentary facies of iron formation. *Econ. Geol.*, **49**, 235–93.

Kanjilal, S. (1966). Geology of the area round the village Kathwar, District Sirmur, H.P. *Unpublished M.Sc. Thesis.*, B.H.U.

Krumbein, W. C., and Pettijohn, F. J. (1938). Manual of Sedimentary Petrography, New York, Appleton-Century.

Krynine, P. D. (1948). The megascopic study and field classification of sedimentary rocks. *J. Geol.*, **56**, 130–65.

Pettijohn, F. J. (1954). Classification of sandstones. *J. Geol.*, **62**, 360–365.

———— (1957). Sedimentary Rocks. Harper and Brothers.

Potter, P. E., and Pettijohn, F. J. (1963). Paleocurrents and Basin Analysis. Springer-Verlag, Berlin.

Powers, M. C. (1953). A new roundness scale for sedimentary particles. *J. sedim. Petrol.*, **23**, 117–19.

Towe, K. M. (1962). Clay mineral digenesis as a possible source of silica cement in sedimentary rock. *J. sedim. Petrol.*, **32**, 26–28.

ESCAPE OF PHOTONS FROM A ROTATING STAR

by R. M. Misra *and* Udit Narain, *Department of Physics, University of Gorakhpur, Gorakhpur*

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In this note the motion of light rays in the stationary gravitational field of a rotating star is discussed. It has been shown that rotation drastically modifies the behaviour of the ray. However, for stars with small angular velocity the escape of ray is similar to a non-rotating case.

1. Introduction

Recently, interest in the problem of gravitational collapse has been revived by the discovery of quasi-stellar radio sources and the possibility of ascribing their enormous energy output to gravitational origin. Under continued gravitational collapse the energy of the star is given away in the form of electromagnetic radiation. It is interesting to study the effect of the gravitational field of the collapsing matter on the escaping radiation. The effect of the gravitational field of a stationary spherically symmetric body on escaping radiation has been extensively discussed by Hagihara (1931), Darwin (1959, 1961) and Synge (1966). In this note we wish to discuss the escape of photons from a spherically symmetric massive body which is rotating about an axis passing through the centre of the body.

The rigorous solution of Einstein's field equations corresponding to the above problem is, unfortunately, not known and, therefore, we make use of an approximation method. We assume that the angular velocity of the rotating body is small and consider the effect of rotation as a small perturbation away from the static Schwarzschild metric. A general perturbation away from the Schwarzschild metric will give rise to various modes of gravitational radiation (Brill and Hartle 1964). Here we confine ourselves to the case when resulting metric is stationary.

2. The Metric and the Path of the Rays

Let the axis of rotation of the body be taken as z-axis of the coordinate system. Then we may take the line element of the external field of the rotating star as (Kerr 1963)

$$ds^2 = \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) - \left(1-\frac{2m}{r}\right) dt^2$$
$$+\frac{2ma}{r^3} r^2 \sin^2\theta\, d\phi \left\{dt+\left(1-\frac{2m}{r}\right)^{-1} dr\right\} \quad .. \quad .. \quad .. \quad (1)$$

Here m is the mass and a is a constant such that ma is the angular momentum of the body. The form of the line element is suggested from the linearized theory. It is seen that (1) differs from Schwarzschild metric only in the addition of the last term. (The metric given in eqn. (1) is, however, an approximation of Kerr's metric.)

It is convenient to introduce the dimensionless coordinates

$$\rho = \frac{r}{2m}, \quad \tau = \frac{t}{2m}$$

so that eqn. (1) reads

$$ds^2 = 4m^2[(1-\rho^{-1})^{-1}\,d\rho^2+\rho^2(d\theta^2+\sin^2\theta\,d\phi^2)-(1-\rho^{-1})\,d\tau^2 + K\rho^{-1}\sin^2\theta\,d\phi\{d\tau+(1-\rho^{-1})^{-1}d\rho\}] \quad .. \quad .. \quad .. \quad .. \quad (2)$$

where $K = a/m$ is a characteristic constant of the rotating body and for a given body of spherical shape depends on its angular velocity. The above equation shows that for all stars without rotation the behaviour of the ray of light may be discussed in a single argument because m appears as a common factor if the last term were absent. Thus we conclude that in case of rotating stars the path of light rays will be determined by their mass as well as angular velocity.

In order to discuss the path of a given light ray we choose a coordinate system in which the ray is emitted initially in the plane $\theta = \pi/2$. Now it has been shown in the Appendix that it will continue to move in the same plane in the approximation we are using here. Then we obtain the equations of motion of the null geodesic as

$$\left.\begin{aligned} &\frac{d\phi}{du} = A\rho^{-2}, \quad \frac{d\tau}{du} = AB(1-\rho^{-1})^{-1} \\ &(1-\rho^{-1})^{-1}\left(\frac{d\rho}{du}\right)^2+\rho^2\left(\frac{d\phi}{du}\right)^2-(1-\rho^{-1})\left(\frac{d\tau}{du}\right)^2 \\ &+K\rho^{-1}\frac{d\phi}{du}\left\{\frac{d\tau}{du}+(1-\rho^{-1})^{-1}\frac{d\rho}{du}\right\} = 0 \end{aligned}\right\} \quad .. \quad .. \quad (3)$$

where A and B are constants depending upon the initial conditions and u is an affine parameter along the ray. The constant B gives the inverse of the distance of the light path from the point of origin of the coordinate system (Bergmann 1960). In order to avoid Schwarzschild singularity we have to discuss the above differential equations for $\rho_0 \leqslant \rho < \infty$ where $1 < \rho_0 = \frac{r_0}{2m}$. Suppose now that we follow a particular ray emanating from the surface of the star. Then, from (3), the differential equation of the ray is obtained as

$$\left(\frac{d\rho}{d\phi}+\tfrac{1}{2}K\rho^{-1}\right)^2 = \rho^4\left[B^2-\frac{\rho-1}{\rho^3}-\frac{KB}{\rho^3}+\frac{K^2}{4\rho^6}\right] \quad .. \quad .. \quad (4)$$

Thus for a star of given gravitational intensity the path of the ray will be given by the above equation.

3. Discussion and Conclusion

To discuss the escape of rays in general we draw curves with equation

$$B^2 - \frac{K}{\rho^3}B - \frac{\rho - 1}{\rho^3} + \frac{K^2}{4\rho^6} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

with B and ρ as coordinates for different values of K. A point in this diagram represents a point on a ray. In Fig. 1 we have drawn curves for $K = 0{\cdot}5$, 1, 1·5, 2. For the sake of comparison we have also drawn the corresponding

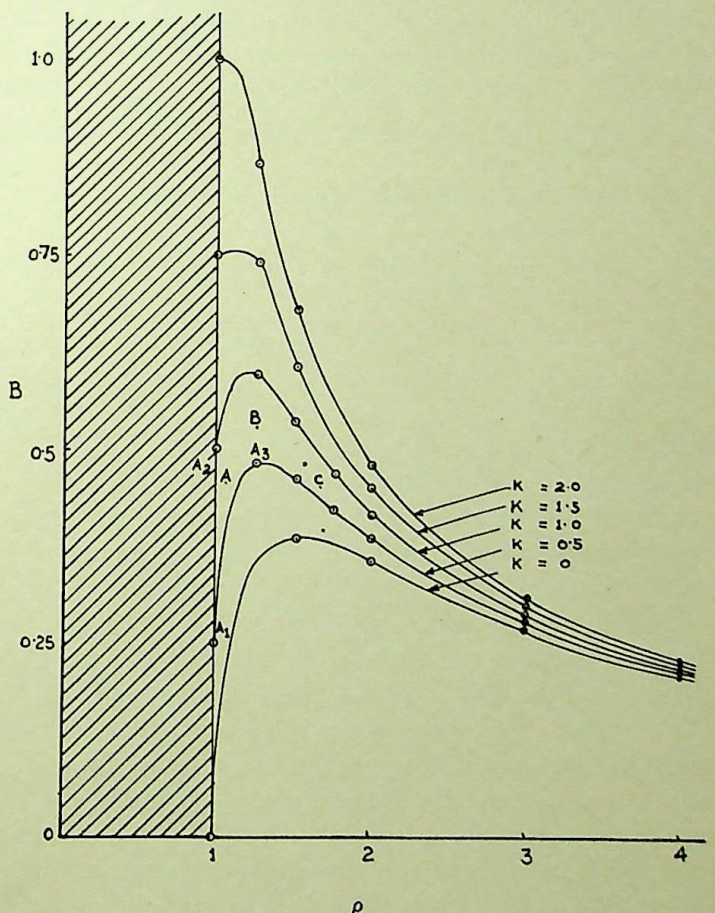

Fig. 1. Escape of photons from a rotating star (curve with $K = 0$ corresponds to a non-rotating star and curves for $K = 0{\cdot}5$, 1·0, 1·5 and 2·0 correspond to stars with increasing angular velocity).

curve for a non-rotating star. Since the right-hand side of eqn. (4) cannot be negative, no representative point can lie below the curve for a given value of

K. Further since ρ lies in the range $1 \leqslant \rho \leqslant \infty$ no point can lie to the left of the line $\rho = 1$ (shown shaded in Fig. 1).

The interesting point is that for small values of K the curves have maxima, similar to the case of non-rotating stars, which shift towards left for increasing values of K. But for $K > 1{\cdot}5$ (approx.) there are no maxima. This leads to the conclusion that as long as the angular velocity of rotation is small, the escape of the ray from a rotating star is similar to that of a stationary star. Thus, three typical points, say A, B and C, may be seen in the diagram for a particular curve, say for $K = 0{\cdot}5$, having a maxima. In each case, as we move along an outgoing ray, the representative point moves horizontally, starting to the right and continuing unless it runs into the curve. For those points, e.g. A, for which the horizontal line runs into the curve the ray has an apse. But if the representative point lies above or to the right of the maximum (e.g. B or C), it may be moved horizontally to infinity without meeting the curve. It is clear from the diagram that the curvilinear triangle A_1 A_2 A_3 is the domain of recapture.

In case of stars with large angular velocity we have a large value of K and corresponding curves have no maxima. It means that none of the rays has an apse and a ray once emitted from the surface of the star escapes to infinity.

The above considerations apply to all rays, whether incoming or outgoing. If we wish to discuss the behaviour of an incoming ray, the representative point will move horizontally from right to left. All the previous arguments equally apply to this case as well.

Thus, we see that the escape of a ray emitted from the surface of a star is sufficiently modified due to rotation. But there are certain limitations to the conclusions which we have drawn here. Firstly, we have used an approximate form of Kerr's exact metric (1963) retaining the first power of the constant a. In doing so we intended the metric to be valid to first order in the angular momentum parameter. But to this order we obtain the Schwarzschild metric plus a term in $d\phi dt$. Although many authors have used this procedure (Brill and Cohen 1966, and Landau and Lifschitz 1962) but there is no reasonable justification for adding a term proportional to $d\phi dt$. The conclusions drawn here are, therefore, not rigorous.* Secondly, the behaviour of the rays in only equatorial plane is discussed.† Thirdly, the constant K in eqn. (2) will generally have lower values than the one we have assumed here.

Nevertheless, the conclusions drawn in this note are interesting. It is expected that more exact treatment of this problem will lead to similar conclusions.

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* Note added in the proof: For the full discussion of Kerr metric, *see* Boyer, R. H., and Lindquist, R. W. *J. Math. Phys. 8*, 265 (1967).

† For a complete discussion all the rays should be discussed.

Acknowledgement

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References

Bergmann, P. G. (1960). Introduction to the Theory of Relativity. Asia Publishing House, Bombay, p. 219.

Brill, D. R., and Cohen, J. M. (1966). Rotating masses and their effect on inertial frames. *Phys. Rev.*, **143**, 1011.

Brill, D. R., and Hartle, J. (1964). Method of the self-consistent field in general relativity and its application to the gravitational geon. *Phys. Rev.*, **135**, B271–B278.

Darwin, C. G. (1959). The gravity field of a particle. *Proc. R. Soc.*, A **249**, 180–194.

———— (1961). The gravity field of a particle (II). *Proc. R. Soc.*, A **263**, 39–50.

Hagihara, Y. (1931). *Jap. J. Astr. Geophys.*, **8**, 67.

Kerr, R. P. (1963). Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, **11**, 237-238.

Landau, L. D., and Lifschitz, E. M. (1962). Classical Theory of Fields. Pergamon Press, London, Sec. 90.

Synge, J. L. (1966). The escape of photons from gravitationally intense stars. *Mon. Not. R. astr. Soc.*, **131**, 463–466.

Appendix

The equation of motion of the ray is obtained from the geodesic equation

$$\frac{d^2x^\mu}{du^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{du}\frac{dx^\beta}{du}=0$$

When the above equation is specialized for the metric (2) for $x^2=\theta$, $x^3=\phi$ and $x^4=t$ we obtain the respective equations for the plane $\theta=\pi/2$

$$\frac{d^2\theta}{du^2}=0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (A1)$$

$$\frac{d^2\phi}{du^2}+\frac{2}{\rho}\frac{d\rho}{du}\frac{d\phi}{du}+\frac{a}{2m\rho^2}\left(\frac{d\phi}{du}\right)^2$$

$$-\frac{a}{4m\rho^5}\left(\frac{d\tau}{du}\right)^2+\left\{\frac{1}{2\rho}(-1-\rho^{-1})^{-2}\right\}\frac{a}{2m\rho^4}\left(\frac{d\rho}{du}\right)^2$$

$$+\frac{a}{2m\rho^4}\left\{\frac{(1-\rho^{-1})^{-1}}{\rho}-1\right\}\frac{d\tau}{du}\frac{d\rho}{du}-\frac{a^2}{4m^2\rho^5}\frac{d\tau}{du}\frac{d\phi}{du}$$

$$+\frac{a^2}{4m^2\rho^5}(1-\rho^{-1})^{-1}\frac{d\rho}{du}\frac{d\phi}{du}=0 \quad .. \quad .. \quad .. \quad .. \quad (A2)$$

$$(1-\rho^{-1})\frac{d^2\tau}{du^2}+\frac{1}{\rho^2}\frac{d\tau}{du}\frac{d\rho}{du}-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}$$

$$+\frac{a^2}{8m^2\rho^6}\left(\frac{d\tau}{du}\right)^2+(1-\rho^{-1})^{-2}\frac{a^2}{4m^2\rho^5}\left(1-\frac{1}{2\rho}\right)\left(\frac{d\rho}{du}\right)^2$$

$$-\frac{a^2}{4m^2\rho^3}\left(\frac{d\phi}{du}\right)^2+\frac{a^2}{4m^2\rho^5}\left\{1-\frac{(1-\rho^{-1})^{-1}}{\rho}\right\}\frac{d\tau}{du}\frac{d\rho}{du}$$

$$+\frac{a^3}{8m^3\rho^6}\frac{d\tau}{du}\frac{d\phi}{du}+\frac{a^3}{8m^3\rho^6}(1-\rho^{-1})^{-1}\frac{d\rho}{du}\frac{d\phi}{du}=0 \quad .. \quad .. \quad (A3)$$

Equation ($A1$) shows that a ray initially moving in the plane $\theta=\pi/2$ will continue to move in the same plane. The other two equations cannot be integrated as such and we, therefore, make use of the approximation referred to earlier. We expand different terms in decreasing powers of ρ and retain terms of the order $O(\rho^{-3})$. Thus, we obtain

$$\frac{d^2\phi}{du^2}+\frac{2}{\rho}\frac{d\rho}{du}\frac{d\phi}{du}+\frac{a}{2m\rho^2}\left(\frac{d\phi}{du}\right)^2=0 \quad .. \quad .. \quad .. \quad (A4)$$

and

$$(1-\rho^{-1})\frac{d^2\tau}{du^2}+\frac{1}{\rho^2}\frac{d\tau}{du}\frac{d\rho}{du}-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}=0 \quad .. \quad .. \quad (A5)$$

Equations ($A4$) and ($A5$) are completely integrable but for the presence of terms $\frac{a}{2m\rho^2}(d\phi/du)^2$ and $-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}$. But as we have already remarked, we consider the effect of rotation as a small perturbation away from the stationary body. The corresponding equations for spherically symmetric stationary body are completely integrable and if we compute the order of magnitude of $\frac{a}{2m\rho^2}\left(\frac{d\phi}{du}\right)^2$ and $-\frac{a}{2m\rho^2}\frac{d\rho}{du}\frac{d\phi}{du}$ this is found to be much smaller quantity than we are interested in. Therefore, we are justified in neglecting even these terms. Thus we obtain eqn. (3).

ON IMPROVING AN APPROXIMATE FACTOR OF A POLYNOMIAL WITH REAL COEFFICIENTS BY AN ITERATIVE PROCESS

by R. D. Bhargava *and* R. N. Soral,* *Indian Institute of Technology, Bombay 76*

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Bairstows' (1914) method of improving an approximate quadratic factor of a polynomial of degree $n \geqslant 2$ with real coefficients is generalized to improve an approximate factor of an arbitrary degree $m < n$. The treatment is an algebraic one but the numbers can be easily substituted. As is easily seen the method can have several applications. The algorithm for such a method is given and its convergence is proved in this paper.

§ 1. Consider the polynomial

$$F(x) = a_0x^n+a_1x^{n-1}+a_2x^{n-2}+ \ldots +a_{n-1}x+a_n$$

of degree n with real coefficients $a_0, a_1, \ldots, a_n$ and let an approximate factor of it be

$$P(x) = x^m+p_1x^{m-1}+p_2x^{m-2}+ \ldots +p_{m-1}x+p_m$$

where $m < n$. Then the problem is to improve the coefficients $p_1, p_2, \ldots, p_m$ of $P(x)$ so that the modified coefficients are better approximation to the factor of $F(x)$ than the original ones. In other words, if the identity

$$F(x) = (x^m+p_1x^{m-1}+ \ldots +p_m)Q(x)+R_1x^{m-1}+R_2x^{m-2}+ \ldots +R_m \qquad (1)$$

holds, where

$$Q(x) = b_0x^{n-m}+b_1x^{n-m-1}+ \ldots +b_{n-m}$$

and the coefficients $b_0, b_1, \ldots, b_{n-m}, R_1, R_2, \ldots, R_m$ depend only upon $p_1, p_2, \ldots, p_m$, then the problem is to find $p_1, p_2, \ldots, p_m$, so that

$$R_i(p_1, p_2, \ldots, p_m) = 0; \quad i = 1, 2, \ldots, m \qquad (2)$$

For this we use Newton's interative process. That is, we find the small corrections $\Delta p_1, \Delta p_2, \ldots, \Delta p_m$, so that

$$R_i(p_1+\Delta p_1, p_2+\Delta p_2, \ldots, p_m+\Delta p_m) = 0; \quad i = 1, 2, \ldots, m \qquad (3)$$

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* R. N. Soral has left for U.S.A. recently.

Expanding eqns. (3) in Taylor series and supposing that the squares and products of Δp_i are negligible as compared to Δp_i, we truncate after the first order terms and get a system of m linear simultaneous equations

$$R_i(p_1, p_2, \ldots, p_m) + \sum_{t=1}^{m} \frac{\partial R_i}{\partial p_t} \Delta p_t = 0; \quad i = 1, 2, \ldots, m \qquad \ldots \quad (4)$$

in m unknowns Δp_1, Δp_2, $\ldots$, Δp_m. When this system has been solved, the procedure is repeated with the corrected values for $p_1, p_2, \ldots, p_m$. The problem, however, is to find the values of $\partial R_t/\partial p_i$. For this, we proceed as follows. We use the identity (1) to obtain

$$b_k = a_k - \sum_{j=1}^{m} b_{k-j}\, p_j \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where

$$k = 0, 1, 2, \ldots, n-m, \quad b_{-1} = b_{-2} = \ldots = b_{-m} = 0$$

and

$$R_i = a_{n-m+i} - \sum_{k=i}^{m} p_k b_{n-m+i-k}$$

$$= b_{n-m+i} + \sum_{k=1}^{i-1} p_k b_{n-m+i-k} \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

where $i = 1, 2, \ldots, m$ and $b_{n-m+1}, b_{n-m+2}, \ldots, b_n$ are also defined by eqn. (5) for $k = n-m+1, \ldots, n$.

Differentiating eqn. (5) with respect to p_t we obtain

$$\frac{\partial b_k}{\partial p_t} = -\left(b_{k-t} + \sum_{j=1}^{m} \frac{\partial b_{k-j}}{\partial p_i}\, p_j\right) \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

It may be noted that a_k are constants and, therefore, their derivatives are zero. Putting $\dfrac{\partial b_k}{\partial p_t} = -c_{k-t}$, it is obvious that

$$\frac{\partial b_{k+1}}{\partial p_1} = \frac{\partial b_{k+2}}{\partial p_2} = \ldots = \frac{\partial b_{k+m}}{\partial p_m} = -c_k$$

for all k. Inserting c_k in (7) we find

$$c_k = b_k - \sum_{j=1}^{m} c_{k-j}\, p_j \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where $k = 0, 1, 2, \ldots, n-m$, $c_{-1} = c_{-2} = \ldots = c_{-m} = 0$. We see that c_k are computed from the b_k exactly as the b_k were computed from the a_k. We

now differentiate eqn. (6) with respect to p_t and write R_{it} for $\partial R_i/\partial p_t$ to obtain

$$R_{it} = \frac{\partial b_{n-m+i}}{\partial p_t} + \sum_{k=1}^{i-1} \left(p_k \frac{\partial b_{n-m+i-k}}{\partial p_t} + b_{n-m+i\ k} \frac{\partial p_k}{\partial p_t} \right)$$

$$= \begin{cases} \sum_{k=i}^{m} p_k c_{n-m+i-t-k}, & \text{if } t < i \\ -\sum_{k=0}^{i-1} p_k c_{n-m+i-t-k} \quad (p_0 = 1) & \text{if } t \geqslant i \end{cases} \qquad \ldots \quad \ldots \quad (9)$$

Thus the values of R_{it} are known and the system (4) can, therefore, be solved for $\Delta p_1, \Delta p_2, \ldots, \Delta p_m$. The whole procedure is summarized in the following algorithm.

Algorithm—Given an approximate factor

$$P(x) = x^m + p_1 x^{m-1} + \ldots + p_m$$

of the polynomial

$$F(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

determine for each $\gamma = 0, 1, 2, \ldots$ a sequence of coefficients $p_k^{(\gamma)}$ $(k = 1, 2, \ldots, m)$ of $P(x)$ as follows. Determine

(i) the sequence $\{b_k^{(\gamma)}\}$ from

$$b_k^{(\gamma)} = a_k - \sum_{j=1}^{m} b_{k-j}^{(\gamma)} p_j^{(\gamma)}$$

where $k = 0, 1, 2, \ldots, n$ and $b_{-1}^{(\gamma)} = b_{-2}^{(\gamma)} = \ldots = b_{-m}^{(\gamma)} = 0$.

(ii) the sequence $\{c_k^{(\gamma)}\}$ from

$$c_k^{(\gamma)} = b_k^{(\gamma)} - \sum_{j=1}^{m} c_{k-j}^{(\gamma)} p_j^{(\gamma)}$$

where $k = 1, 2, \ldots, n-m$ and $c_{-1}^{(\gamma)} = c_{-2}^{(\gamma)} = \ldots = c_{-m}^{(\gamma)} = 0$.

(iii) the sequence $\{R_i^{(\gamma)}\}$ from

$$R_i^{(\gamma)} = b_{n-m+i}^{(\gamma)} + \sum_{k=1}^{i-1} p_k^{(\gamma)} b_{n-m+i-k}^{(\gamma)}$$

where $i = 1, 2, \ldots, m$.

(iv) the sequence $\{R_{it}^{(\gamma)}\}$ from

$$R_{it}^{(\gamma)} = \begin{cases} \sum_{k=i}^{m} p_k^{(\gamma)} c_{n-m+i-t-k}^{(\gamma)} & \text{if } t < i \\ -\sum_{k=0}^{i-1} p_k^{(\gamma)} c_{n-m+i-t-k}^{(\gamma)} & \text{if } t \geqslant i \end{cases}$$

where $i, t = 1, 2, \ldots, m$. Then

(v) the set $\{p_k^{(\gamma+1)};\ k = 1, 2, \ldots, m\}$ is obtained by $p_k^{(\gamma+1)} = p_k^{(\gamma)} + \Delta p_k^{(\gamma)}$, where $\{\Delta p_k^{(\gamma)};\ k = 1, 2, \ldots, m\}$ is determined by

$$\sum_{t=1}^{m} R_{it}^{(\gamma)} \Delta p_t^{(\gamma)} = -R; \quad i = 1, 2, \ldots, m.$$

§ 2. To prove the convergence, we give below the conditions sufficient to imply that there exists a neighbourhood around a real solution $p = (p_1, p_2, \ldots, p_m)$ of the system (2) such that for an initial choice $p^{(0)} = (p_1^{(0)}, p_2^{(0)}, \ldots, p_m^{(0)})$ in that neighbourhood, the above process converges to the solution.

Let $D(p)$ denote the determinant of the coefficient matrix of system (4), i.e.

$$D(p) = \begin{vmatrix} R_{11} & R_{12} & \ldots & R_{1m} \\ R_{21} & R_{22} & \ldots & R_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ R_{m1} & R_{m2} & \ldots & R_{mm} \end{vmatrix}$$

and let $(s_1, s_2, \ldots, s_m) = s$ be a solution of the system (2). If the function $R_i(p)$ have continuous first partial derivatives in the neighbourhood of the solution $p = s$, and if the determinant $D(p)$ is non-vanishing in the neighbourhood of this solution, then Newton's process of solving the system (2) converges (Householder 1953). Also if $D(s) \neq 0$, then since $D(p)$ is continuous, it follows that $D(p) \neq 0$ for all points p sufficiently close to s. Thus in order to show the uniqueness of the solution of the system (4) and the convergence of the above method we have to show that $D(s) \neq 0$. To prove this for general case seems to be difficult but if the zeros $x_1, x_2, \ldots, x_m$ of the factor $x^m + s_1 x^{m-1} + \ldots + s_m$ of $F(x)$ are simple and distinct the result can be proved as follows. Since $F(x)$ is independent of $p_1, p_2, \ldots, p_m$, differentiating the identity (1) with respect to $p_1, p_2, \ldots, p_m$ respectively, we obtain

$$0 = x^{m-1}Q(x) + P(x)\frac{\partial Q}{\partial p_1} + x^{m-1}R_{11} + x^{m-2}R_{21} + \ldots + R_{m1}$$

$$0 = x^{m-2}Q(x) + P(x)\frac{\partial Q}{\partial p_2} + x^{m-1}R_{12} + x^{m-2}R_{22} + \ldots + R_{m2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$0 = Q(x) + P(x)\frac{\partial Q}{\partial p_m} + x^{m-1}R_{1m} + x^{m-2}R_{2m} + \ldots + R_{mm}$$

Setting $p_i = s_i$ and $x = x_i$ $(i = 1, 2, \ldots, m)$ we obtain m systems

$$\left.\begin{aligned} x_i^{m-1}R_{11} + x_i^{m-2}R_{21} + \ldots + R_{m1} &= -x_i^{m-1}Q_i \quad (i = 1, 2, \ldots, m) \\ x_i^{m-1}R_{12} + x_i^{m-2}R_{22} + \ldots + R_{m2} &= -x_i^{m-2}Q_i \quad (i = 1, 2, \ldots, m) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ x_i^{m-1}R_{1m} + x_i^{m-2}R_{2m} + \ldots + R_{mm} &= -Q_i \quad (i = 1, 2, \ldots, m) \end{aligned}\right\} \quad (10)$$

each of m linear simultaneous equations in m unknowns with the non-vanishing determinant

$$|A| = \begin{vmatrix} x_1^{m-1} & x_1^{m-2} & \dots & 1 \\ x_2^{m-1} & x_2^{m-2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ x_m^{m-1} & x_m^{m-2} & \dots & 1 \end{vmatrix}$$

$$= \prod_{i>j} (x_i - x_j), \quad 1 \leqslant j < i \leqslant m$$

$\neq 0$ (because the zeros $x_1, x_2, \dots, x_m$ of $F(x)$ are distinct)

of the coefficient matrix A. Here $Q_i = Q(x_i)$ and the derivatives R_{ij} are taken at $p = s$. Now we shall show that the solution vectors of the systems in system (10) are linearly independent. For this we have to show that

$$a_1(R_{11}, R_{21}, \dots, R_{m1}) + a_2(R_{12}, R_{22}, \dots, R_{m2}) + \dots + a_m(R_{1m}, R_{2m}, \dots, R_{mm}) = 0 \quad \dots (11)$$

is true only for $a_1 = a_2 = \dots = a_m = 0$. Let the identity (11) be true of any constants $a_1, a_2, \dots, a_m$. Then

$$\left.\begin{aligned} a_1R_{11} + a_2R_{12} + \dots + a_mR_{1m} &= 0 \\ a_1R_{21} + a_2R_{22} + \dots + a_mR_{2m} &= 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots & \\ a_1R_{m1} + a_2R_{m2} + \dots + a_mR_{mm} &= 0 \end{aligned}\right\} \quad \dots \quad \dots (12)$$

From (10) and (12) we obtain

$$x_i^{m-1}Q_ia_1 + x_i^{m-2}Q_ia_2 + \dots + Q_ia_m = 0 \quad (i = 1, 2, \dots, m) \quad \dots (13)$$

But $Q_i = Q(x_i) \neq 0$, because the zeros $x_1, x_2, \dots, x_m$ of $P(x)$ are simple zeros of $F(x)$.

Dividing (13) by Q_i we obtain

$$x_i^{m-1}a_1 + x_i^{m-2}a_2 + \dots + a_m = 0 \quad (i = 1, 2, \dots, m) \quad \dots (14)$$

Since the determinant $|A|$ of the coefficient matrix of the above system (14) is non-zero, this system has only trivial solution, i.e. $a_1 = a_2 = \dots = a_m = 0$. Hence the solution vectors $(R_{11}, R_{21}, \dots, R_{m1})$, $(R_{12}, R_{22}, \dots, R_{m2}), \dots,$ $(R_{1m}, R_{2m}, \dots, R_{mm})$ are linearly independent from which it follows that

$$D(s) = \begin{vmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ R_{21} & R_{22} & \dots & R_{2m} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ R_{m1} & R_{m2} & \dots & R_{mm} \end{vmatrix} \neq 0$$

Thus for all p sufficiently close to s the system (4) has a unique solution $\Delta p_1, \Delta p_2, \ldots, \Delta p_m$ and the point $(p_1+\Delta p_1, p_2+\Delta p_2, \ldots, p_m+\Delta p_m)$ is more close to s than p.

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REFERENCES

Bairstows, L. (1914). Investigations relating to the stability of the aeroplane. *Reports and Memoranda No. 154 of Advisory Committee for Aeronautics.*

Householder, A. S. (1953). Principles of Numerical Analysis. McGraw-Hill Book Co. Inc., New York.

ON INITIAL DEVELOPMENT OF AXISYMMETRIC WAVES IN FLUIDS OF FINITE DEPTH

by L. DEBNATH,* *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge*

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An initial value approach to the axisymmetric wave problem in a fluid of limited and very shallow depth is presented. An asymptotic solution of the problem has been worked out in a simple and rigorous way by the similar method developed by the author (Debnath 1969) in an earlier paper. Finally, the principal behaviour of the axisymmetric wave motions has been discussed in considerable detail and some physical arguments has been put forward in order to explain the strange character of the solution related to the concentrated pressure distribution with the forcing frequency ω.

1. INTRODUCTION

Debnath (1969) has suggested various reasons in favour of the initial value investigation into the general wave problems and explained the relation between the steady state and properly posed initial value problem in considerable detail. Usually, it is assumed that the solution of the steady state problem obtained by imposing a radiation condition at infinity is generally the limit as time $T \to \infty$ of the solution derived by considering a well-posed initial value problem. Nevertheless, there appears to be no generally rigorous proof that this is, in fact, the case and it is usually necessary to examine the limiting procedure in each individual configuration. In the same paper, the author has introduced a linearized theory of axisymmetric wave motions in a fluid of infinite depth due to local disturbances with the forcing frequency ω acting on the undisturbed free surface of the fluid. The asymptotic solution of the initial value problem has been derived in a simple and rigorous manner. Finally, the principal features of the wave motions have been described in some detail.

In this connection, reference may be made to the original works of Stoker (1957) who considered the unsteady two-dimensional surface wave problem due to a periodic concentrated pressure distribution. With due investigation, he suggested that steady state wave problem with the Sommerfeld radiation

* *Present address:* Mathematics Department, P.O. Box 2754, East Carolina University, Greenville, North Carolina 27834, U.S.A.

condition at infinity should be derivable as the limit of the corresponding initial value problem.

The corresponding axisymmetric wave problem in a fluid of limited and very shallow depth is also interesting and important on its own merit, hence it deserves a special attention. So far, no such work has been reported to the current literature at least available to the author.

The object of this discussion is to present a linearized theory of initial development of axisymmetric surface waves in an inviscid, incompressible and homogeneous fluid of limited and very shallow depth generated by an arbitrary periodic disturbance with the forcing frequency ω acting on the undisturbed free surface of the fluid. An asymptotic solution of the problem has been derived for sufficiently large times by the similar method developed by the author (Debnath 1969) in his earlier paper. Finally, the limiting behaviour of the solution has been examined in some detail and the main features of the wave motions have been predicted. This is followed by some physical arguments in favour of the strange behaviour of the solution related to the oscillating point-pressure distribution with the forcing frequency ω.

2. Mathematical Formulation of the Problem

We consider the axisymmetric wave problem in an inviscid, incompressible and homogeneous fluid of finite depth due to a periodic surface pressure distribution in the form

$$\left.\begin{aligned} P(R, T) &= P(R)\, e^{i\omega T} H(T), \quad R \leqslant A \\ &= 0 \qquad\qquad\quad , \quad R > A \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (2.1)$$

acting on the free surface $Y = 0$ of the fluid initially at rest, where $P(R)$ is an arbitrary function of R, ω is the fixed frequency and $H(T)$ is the Heaviside unit function of time T.

We assume that X-Z plane be the undisturbed horizontal free surface and Y-axis be vertical positive upwards. We take cylindrical polar coordinates (R, Θ, Y) with cylindrical symmetry about the Y-axis such that R is equal to $\sqrt{X^2+Z^2}$ from the Y-axis and the origin of coordinate is taken on the free surface.

In view of the fact that the motion starts from rest, there exists a velocity potential $\Phi(R, Y; T)$ which is governed by the Laplace equation

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial R^2}+\frac{1}{R}\frac{\partial\Phi}{\partial R}+\frac{\partial^2\Phi}{\partial Y^2} = 0 \quad .. \quad .. \quad .. \quad (2.2)$$

$$-h \leqslant Y \leqslant 0, \; 0 \leqslant R < \infty$$

where h denotes the depth of the fluid.

The condition at the bottom surface is

$$\Phi_Y = 0 \quad \text{on } Y = -h \quad .. \quad .. \quad .. \quad .. \quad (2.3)$$

Neglecting surface tension, the linearized free surface boundary conditions are given by

$$\left.\begin{aligned} &\Phi_T + gE = -\frac{1}{\rho}P(R,T) \\ &\Phi_Y = E_T \end{aligned}\right\} \quad \begin{matrix} Y=0 \\ T>0 \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} (2.4) \\ (2.5) \end{matrix}$$

where suffixes denote partial differentiation and $E = E(R, T)$ represents the vertical surface displacement, ρ the density and g the acceleration due to gravity.

The initial conditions are given by

$$\left.\begin{aligned} &\Phi = E = 0 \quad \text{at } T = 0 \\ &[\Phi_T(R, 0; T)]_{T=0} = -\frac{1}{\rho}P(R, 0) \end{aligned}\right\} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} (2.6) \end{matrix}$$

We further assume that the functions $\Phi(R, Y; T)$ and $E(R, T)$ possess the Hankel transform with respect to R in the generalized sense.

Remarks: The problem formulated above can be investigated as a steady state just by omitting the initial conditions (2.6). For the steady state analysis, one has to impose Sommerfeld's radiation condition at infinity which is, in fact, essential in order to derive a solution of physical interest.

3. Solution of the Problem

It is convenient to introduce dimensionless variables r, x, y, z, a, t, η and ϕ defined by the following relations:

$$(r, x, y, z, a) = \frac{\omega^2}{g}(R, X, Y, Z, A), \quad t = \omega T, \quad \eta = \frac{P\omega^4}{\rho g^3}E,$$

and

$$\phi = \frac{P\omega^5}{\rho g^4}\Phi,$$

and we introduce a non-dimensional parameter D defined by $D = \omega^2 h/g$. Substituting these relations into the basic eqns. (2.2)–(2.6), we obtain

$$\phi_{rr} + \frac{1}{r}\phi_r + \phi_{yy} = 0 \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} (3.1) \end{matrix}$$

$$-D \leqslant y \leqslant 0, \ 0 \leqslant r < \infty$$

$$\phi_y = 0 \quad \text{on } y = -D \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} (3.2) \end{matrix}$$

$$\left.\begin{aligned} &\phi_t + \eta = -p(r, t) \\ &\phi_y = \eta_t \end{aligned}\right\} \quad \begin{matrix} y = 0 \\ t > 0 \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} (3.3) \\ (3.4) \end{matrix}$$

$$\left.\begin{aligned} &\phi = \eta = 0 \quad \text{at } t = 0 \\ &\phi_t = -p(r, t) \quad \text{at } t = 0, \ y = 0 \end{aligned}\right\} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} \\ \end{matrix} \qquad \begin{matrix} (3.5) \end{matrix}$$

As in our earlier paper (Debnath and Rosenblat *in press*), we introduce the Laplace transform (Tranter 1966) $\bar{\phi}$, $\bar{\eta}$ of the functions ϕ, η respectively

with respect to t and then we introduce the Hankel transform $\bar{\bar{\phi}}$, $\bar{\bar{\eta}}$ (Lighthill 1958) of the functions $\bar{\phi}$, $\bar{\eta}$ respectively with respect to r. This enables us to simplify eqns. (3.1)–(3.5) and to write down the solutions for $\bar{\bar{\phi}}$ and $\bar{\bar{\eta}}$ without any difficulty in the form

$$\bar{\bar{\phi}}(k, y; s) = -\frac{s\,\bar{\bar{p}}(k, s)\cosh k(y+D)}{(s^2+\alpha^2)\cosh kD} \quad .. \quad .. \quad .. \quad (3.6)$$

$$\bar{\bar{\eta}}(k, s) = -\frac{\alpha^2(k)\,\bar{\bar{p}}(k, s)}{(s^2+\alpha^2)} \quad .. \quad .. \quad .. \quad .. \quad (3.7)$$

where α^2 is given by the relation

$$\alpha^2 = \alpha^2(k) = k \tanh kD. \quad .. \quad .. \quad .. \quad (3.8)$$

Making reference to the inversion formula for the Laplace transform combined with its convolution theorem and then to the inverse theorem of Hankel transformation, we readily obtain

$$\phi(r, y; t) = \int_0^\infty (i\cos \alpha t - \alpha \sin \alpha t - ie^{it})\, k\,\bar{p}(k) J_0(kr)\frac{\cosh k(y+D)}{(\alpha^2-1)\cosh kD}\,dk \quad .. \quad (3.9)$$

$$\eta(r, t) = \int_0^\infty (i\sin \alpha t + \alpha\cos \alpha t - \alpha e^{it})\, k\,\alpha(k)(\alpha^2-1)^{-1}\,\bar{p}(k) J_0(kr)\,dk. \quad .. \quad (3.10)$$

The above integrals (3.9) and (3.10) are the exact solution of the problem. But, in general, they cannot be worked out exactly. Hence asymptotic methods are utilized to evaluate them for a deeper understanding of the transient wave motions.

In the case of infinite depth of fluids, i.e. when $D \to \infty$, the integral for the velocity potential $\phi(r, y; t)$ as well as the surface elevation $\eta(r, t)$ has the following form:

$$\phi(r, y; t) = \int_0^\infty (i\cos \sqrt{k}t - \sqrt{k}\sin \sqrt{k}t - ie^{it}) e^{ky}\, k\,\bar{p}(k) J_0(kr)(k-1)^{-1}\,dk \quad .. \quad (3.11)$$

$$\eta(r, t) = \int_0^\infty (\sqrt{k}\cos \sqrt{k}t + i\sin \sqrt{k}t - \sqrt{k}e^{it})\, k^{3/2}\,\bar{p}(k) J_0(kr)(k-1)^{-1}\,dk. \quad .. \quad (3.12)$$

These results are exactly identical with those already obtained by Debnath (1969).

4. Asymptotic Analysis of the Solution

As already mentioned that the function $p(r)$ involved in the non-dimensional pressure distribution may be arbitrary, hence it would be sufficient for the determination of the principal behaviour of the wave motions to take the simple non-dimensional form as

$$p(r, t) = e^{it}\,p(r), \quad r \leqslant a$$
$$= 0 \quad , \quad r < a$$

with the prescribed form of $p(r)$ as

$$\text{(a)}\quad p(r) = \delta(r)/r$$

where $\delta(r)$ is the Dirac function of distribution and

$$\text{(b)}\quad p(r) = 1$$

$$\text{(c)}\quad p(r) = \exp\left(-r^2/r_0^2\right)$$

$$\text{(d)}\quad p(r) = J_0(\lambda r)$$

$$\text{(e)}\quad p(r) = (a^2-r^2)^n, \quad n > -1.$$

We proceed to find out an asymptotic representation for the surface elevation $\eta(r, t)$ in the general form and the above particular cases (a)–(e) can be used in the end.

Let us consider the integral (3.10) for the surface elevation $\eta(r, t)$ which is given by

$$\eta(r, t) = \int_0^\infty (i \sin \alpha t + \alpha \cos \alpha t - \alpha e^{it})(\alpha^2-1)^{-1} k\, \alpha(k)\, \bar{p}(k) J_0(kr)\, dk.$$

It may be noticed that this integral has no singularities on $(0, \infty)$, therefore, its path of integration can be deformed into a path M (say) in the complex $s = k+i\mu$ plane as indicated in Fig. 1. The path M coincides with the real axis $(0, \infty)$ except that it is diverted round the zeros of the denominator of the integrand. We can then write down the integral as a sum of two integrals which are now singular at the zeros of (α^2-1).

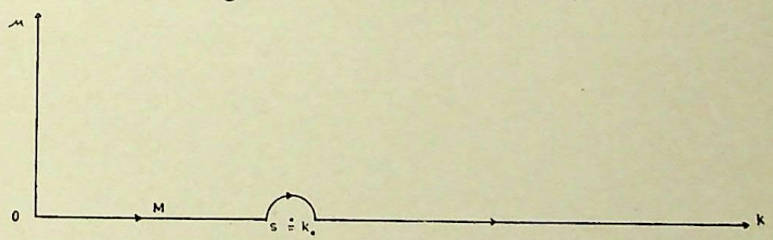

FIG. 1. The complex $s = k+i\mu$.

Hence

$$\eta(r, t) = \int_M (i \sin \alpha t + \alpha \cos \alpha t - \alpha e^{it})\, s\, \alpha(s)(\alpha^2-1)^{-1}\, \bar{p}(s) J_0(sr)\, ds \quad \ldots \quad (4.1)$$

where $\alpha = \alpha(s) = \sqrt{s \tanh sD}$ and $s = k_0$ is the only real root in $(0, \infty)$ of the equation $\alpha^2(s) = 1$.

We thus obtain

$$\eta(r, t) = (I_1 - I_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

where I_1 and I_2 are given by the integrals in the form

$$I_1 = \int_M \bar{p}(s)\, s\, \alpha(s)\, (\alpha^2-1)^{-1}(i \sin \alpha t + \alpha \cos \alpha t) J_0(sr)\, ds$$

$$I_2 = e^{it} \int_M \frac{\bar{p}(s)\, s\, \alpha^2(s) J_0(sr)\, ds}{(\alpha^2-1)}$$

where $-\pi < \arg s \leqslant \pi$.

It may be noticed that the wave integrals I_1 and I_2 obtained above are very much similar in form obtained earlier (Debnath 1969), therefore, similar methods, as before, with a slight modification can be exploited to evaluate them without any difficulty.

Making reference to the method used (Debnath 1969) for the evaluation of the steady state wave integral, we readily write down the result for I_2 in the form

$$I_2 \sim -\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \left[\frac{i\bar{p}(k_0)\, k_0\, \alpha^2(k_0)}{W'(k_0)}\right] e^{i\left(t-rk_0+\frac{\pi}{4}\right)} \quad .. \quad .. \quad (4.3)$$

where the function $W(k)$ is given by

$$W(k) = \alpha^2(k) - 1 \quad .. \quad .. \quad .. \quad .. \quad (4.4)$$

Remarks: It can be easily seen that in the case of infinite depth (i.e. $D \to \infty$), the pole $s = k_0 = 1$ and, consequently, the value of I_2 obtained above is in perfect agreement with the result of Debnath (1969).

In order to evaluate the transient integral I_1, we first replace the Bessel function $J_0(sr)$ by its standard integral formula and next from a simple rearrangement, it easily follows that

$$I_1 = \frac{1}{2\pi} \int_0^{\pi/2} (L_1 + L_2 + L_3 + L_4)\, d\theta \quad .. \quad .. \quad .. \quad (4.5)$$

where the integrals L_1, L_2, L_3 and L_4 are given by

$$L_1 = \int_M \frac{s\,\alpha(s)\bar{p}(s)}{\alpha(s)+1} e^{-itf_-(s)}\, ds$$

$$L_2 = \int_M \frac{\bar{p}(s)\, s\,\alpha(s)}{\alpha(s)-1} e^{itf_-(s)}\, ds$$

$$L_3 = \int_M \frac{\bar{p}(s)\, s\,\alpha(s)}{\alpha(s)-1} e^{itf_+(s)}\, ds$$

$$L_4 = \int_M \frac{\bar{p}(s)\, s\,\alpha(s)}{\alpha(s)+1} e^{-if_+(s)t}\, ds$$

with

$$f_\pm(s) = \left(\alpha \pm \frac{r}{t} s\cos\theta\right).$$

It may be observed that these integrals are similar in nature to those already obtained earlier by Debnath (1968, 1969), Debnath and Rosenblat (*in press*). Therefore, they can be evaluated exactly in a similar way as before with a little modification. However, a few important points may be outlined here so that one can easily understand basic ideas behind the asymptotic evaluation of these integrals stated above for large values of t. Making reference to the works cited above, it is easy to show that the integrals L_3 and L_4 are $O\left(\frac{1}{t}\right)$.

And the stationary (or saddle) point and pole related to the integrals L_1 and L_2 are respectively at $s = k_1$ and $s = k_0$. In fact, $s = k_1$ is the root of the equation

$$f'_-(s) = 0.$$

A similar calculation, as before, leads us to the asymptotic answer to the integrals L_1 and L_2. Then substituting their values to the θ-integral (4.5) and making appeal to the method of stationary phase (Jeffreys and Jeffreys 1946) for large t in order to evaluate the θ-integral, it turns out that

$$I_1 \sim \begin{bmatrix} -\dfrac{\bar{p}(k_0) i k_0\, \alpha(k_0)\{\alpha(k_0)+1\}}{2W'(k_0)} \left(\dfrac{2\pi}{rk_0}\right)^{\frac{1}{2}} e^{i\left\{t\,\alpha(k_0) - rk_0 + \frac{\pi}{4}\right\}}, & k_1 < k_0 \\ 0, & k_1 > k_0 \end{bmatrix}$$

$$+ k_1 \alpha(k_1) \frac{\bar{p}(k_1)}{\left\{4rt \left| f''_-(k_1) \right|\right\}^{\frac{1}{2}}} \left[\frac{e^{i\{k_1 r - t\,\alpha(k_1)\}}}{\alpha(k_1)+1} + \frac{e^{-i\{k_1 r - t\,\alpha(k_1)\}}}{\alpha(k_1)-1}\right] + O\left(\frac{1}{t^2}\right). \quad .. \quad (4.6)$$

Finally, combining the integrals I_1 and I_2 together, we arrive at the asymptotic representation for the surface elevation $\eta(r, t)$ in the form

$$\eta(r, t) \sim \frac{i\bar{p}(k_0) k_0\, \alpha^2(k_0)}{W'(k_0)} \left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} e^{i\left(t - rk_0 + \frac{\pi}{4}\right)} \begin{bmatrix} 1, & k_1 > k_0 \\ 1 - \frac{1}{2}\{\alpha(k_0)+1\}, & k_1 < k_0 \end{bmatrix}$$

$$+ \frac{k_1 \bar{p}(k_1) \alpha(k_1)}{\left\{4r\, t\, k_1 \left| f''_-(k_1) \right|\right\}^{\frac{1}{2}}} \left[\frac{e^{i\{rk_1 - t\,\alpha(k_1)\}}}{\alpha(k_1)+1} + \frac{e^{-i\{rk_1 - t\,\alpha(k_1)\}}}{\alpha(k_1)-1}\right] + O\left(\frac{1}{t^2}\right) \quad (4.7)$$

Remarks: The quantity $k_1 = k_0$ renders the solution (4.7) invalid. So, a special method is needed to obtain a valid solution for $\eta(r, t)$ near the point $k_1 = k_0$. Since our range of interest is large values of t, it seems to us that the derivation of the solution at the above point is not so important here.

5. Asymptotic Solution in the case of Unlimited Depth

In the case of infinite depth of fluid, that is when the parameter $D \to \infty$, the function $\alpha^2(s)$ tends to the limit s. Hence the pole $s = k_0 = 1$ and the saddle or stationary point $s = k_1 = \left(\dfrac{t}{2r}\right)^2$ (Copson 1965) which are in perfect agreement with our previous analysis (Debnath 1969). Therefore, the above solution takes the following form:

$$\eta(r, t) \sim \begin{bmatrix} i\bar{p}(1)\left(\dfrac{2\pi}{r}\right)^{\frac{1}{2}} e^{i\left(t - r + \frac{\pi}{4}\right)}, & t \gg 2r \\ 0, & t \ll 2r \end{bmatrix}$$

$$+ \frac{t^3\, \bar{p}\left(\dfrac{t^2}{4r^2}\right)}{4\sqrt{2}r^4\left(\dfrac{t^2}{4r^2} - 1\right)} \left\{\frac{t}{2r} \cos\left(\frac{t^2}{4r}\right) + i \sin\left(\frac{t^2}{4r}\right)\right\} + O\left(\frac{1}{t^2}\right) \quad .. \quad (5.1)$$

This result is in perfect agreement with those obtained by Debnath (1969) in the particular cases (a)–(d) stated before. In order to compare this result (5.1) with those derived in the earlier paper, one has to substitute the value for the function $\bar{p}(k_1)$.

We would like to mention here the Hankel transform $\bar{p}(k)$ of the function $p(r)$ related to cases (a)–(e). The functions $\bar{p}(k)$ associated with the cases (a)–(e) are respectively given by

$$1,\ \frac{a}{k}J_1(ak),\ 2m^2e^{-k^2m^2},\ \left[m^2=\left(\frac{r_0}{2}\right)^2\right];\quad \int_0^a x\,J_0(\lambda x)J_0(kx)\,dx,$$

and

$$2^n\Gamma(n+1)(ak)^{-(n+1)}a^{2(n+1)}J_{n+1}(ak),$$

where $J_n(x)$ are the Bessel function of the first kind of order n, and the finite integral is a standard one (Watson 1922; Sneddon 1951).

Thus the solution of the problem is explicitly known in both finite and infinite depth cases. And for infinite depth case, we can refer to our earlier paper (Debnath 1969). In the case of limited depth, we intend to summarize our conclusion later on.

6. Axisymmetric Wave Motions in Shallow Fluids

In the other extreme case, that is when $D \ll 1$, we can replace $\alpha^2(s) \equiv s\tanh sD$ by s^2D to the first approximation. Consequently, the integrals involved in the vertical surface elevation $\eta(r, t)$ are simpler in nature. And a procedure similar to that advanced in the papers (Debnath 1968, 1969) can be adopted to obtain the asymptotic solution for $\eta(r, t)$ in the following form:

$$\eta(r,t)\sim\begin{bmatrix} i\left(\dfrac{2\pi}{r\sqrt{D}}\right)^{\frac{1}{2}}\bar{p}\left(\dfrac{1}{\sqrt{D}}\right)e^{i\left(t-\frac{r}{\sqrt{D}}+\frac{\pi}{4}\right)}, & t\gg\dfrac{r}{\sqrt{D}} \\ 0, & t\ll\dfrac{r}{\sqrt{D}}\end{bmatrix} \qquad \ldots\ (6.1)$$

This solution for $\eta(r, t)$ suggests that the waves are advancing with the phase velocity $\sqrt{gh}$ which is independent of the wavelength and the amplitude of the waves decays like $r^{-1/2}$. Evidently, the waves are non-dispersive in character.

7. Discussion and Conclusions

From the salient and inherent features of the properly posed initial value formulation, it follows that the axisymmetric wave problem investigated above achieves a unique solution of physical interest. An asymptotic analysis of the problem enables us to derive explicitly both the steady state and transient wave solutions. Hence solution (4.7) as well as (5.1) for the surface elevation $\eta(r, t)$ describes the behaviour of wave formation at all times instead

of only in the limit. And the solution for the two extreme limiting cases (i) $h \to \infty$ and (ii) $h \to 0$ has also been discussed in some detail.

The above analysis reveals that the transient component of the asymptotic solution associated with cases (b)–(e) does die out very rapidly in the limit $t \to \infty$, for fixed values of r. Therefore, the ultimate steady state is attained in the limit. In other words, the steady state is set up in the limit provided the applied pressure with the forcing frequency ω is distributed over a 'finite' circular region, however small, of the free surface of the fluid. On the other hand, classical steady state is not attained when the harmonic pressure distribution concentrated in a single point of the free surface. These conclusions are in excellent agreement with those obtained earlier in connection with the axisymmetric wave problem in a fluid of infinite depth and the two-dimensional wave problem (Debnath 1968). Although this disagreement is strange to some extent, but not really unexpected for mathematical and physical reasons which have been suggested earlier (1969) and would be stated below.

In all the above cases (b)–(e), the classical steady state solution of special interest is obtained in the form

$$\eta(r, t) \sim i\left(\frac{A}{\sqrt{r}}\right) e^{i\left(t - rk_0 + \frac{\pi}{4}\right)},$$

where A is some known constant depending on the actual functional form of $\bar{p}(k)$ involved in the cases under consideration.

This solution represents the axisymmetric surface waves propagating in the medium with the amplitude decaying like $r^{-1/2}$. And in the limit $D \to \infty$, that is $h \to \infty$, this solution confirms our previous solution obtained earlier (1969).

The author (1969) has suggested some mathematical arguments to explain the strange behaviour of the asymptotic solution (5.1) for the case of the pressure distribution concentrated in a single point with the forcing frequency ω. From the physical point of view, the following argument may be put forward to resolve the strange situation. The sudden harmonic point pressure with the high frequency content generates a continuous spectrum of circular waves of all possible wavelengths and the continuous spectrum is very rich in short wavelength content indeed. Thus the sudden onset and the enrichment of short wavelength content are both needed if there is to be sufficient energy distributed among the slowly varying travelling wave trains so that more and more intense accumulations occur at fixed (large) distance r as the time t approaches to infinity. That is why the transient component of the asymptotic solution (5.1) related to the delta function pressure increases like $O(t^2)$. So the classical steady state is not attained in the limit.

On the contrary, the situation is well behaved when the pulsating pressure is distributed over a finite region of the free surface and, in particular,

when the Gaussian representation of the delta function in the form

$$\delta(r) = \lim_{a \to 0} \frac{1}{a\sqrt{\pi}} e^{-r^2/a^2}$$

is used as a wave-generating mechanism and the limit $t \to \infty$ is taken first.

Therefore, in any physically realistic situation where the surface pressure is distributed over any finite region of non-zero dimension, the ultimate steady state would be attained in the limit and the attainment of the steady state is directly attributable to the double limit process $a \to 0$; $t \to \infty$ in the proper order as suggested earlier and a similar but appropriate energy argument as outlined above.

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References

Copson, E. T. (1965). Asymptotic Expansions. Cambridge University Press, Cambridge.

Debnath, L. (1968). On transient development of surface waves. *Z. angew Math. Phys.*, **19** (Fasc. 6), 948–961.

——— (1969). On transient development of axisymmetric waves. *Proc. natn. Inst. Sci. India*, **35** A, 567–585.

Debnath, L., and Rosenblat, S. The development of waves on a running stream. *Q. Jl Mech. appl. Math.* (*In press*).

Jeffreys, H., and Jeffreys, B. (1946). Methods of Mathematical Physics. Cambridge University Press, Cambridge.

Lighthill, M. J. (1958). An Introduction to Fourier Analysis and Generalized Functions. Cambridge University Press, Cambridge.

Sneddon, I. N. (1951). Fourier Transforms. McGraw-Hill Book Co. Inc., New York.

Stoker, J. J. (1957). Water Waves. Interscience Publishers Inc., New York.

Tranter, C. J. (1966). Integral Transforms in Mathematical Physics. Methuen & Co. Ltd., London.

Watson, G. N. (1922). A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge.

OCCURRENCE OF AUTHIGENIC TOURMALINES IN THE LOWER KALADGI QUARTZITIC SANDSTONES OF SAUNDATTI, MYSORE STATE

by B. V. Govinda Rajulu, *Department of Geology, University of Mysore, Mysore*
and
P. N. Satish, *Mineralogical Institute, University of Mysore, Mysore*

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This paper deals with the morphology, origin and significance of the different varieties of tourmaline, occurring in the Lower Kaladgi quartzitic sandstones outcropping around Saundatti and exhibiting authigenic growths. It is concluded that the diagenetic changes resulting in the growth of minerals belonging to the complex isomorphous series of the tourmaline group are controlled by neither the chemical composition of the detrital nor the host tourmaline grain.

Introduction

The Lower Kaladgi (Pre-Cambrian) formations of Saundatti (lat. 15° 46′, long. 75° 7′), Mysore State, comprise of quartzitic sandstones, shales and limestones resting unconformably over the basement granites. The general classification and their stratigraphical position are as follows:

Basalts	Deccan Traps
Sandstones	Badamis
Limestone Shales Pebbly quartzitic sandstone Grey quartzitic sandstone Pink quartzitic sandstone Basal conglomerate	Kaladgis (Pre-Cambrian)
Granites, dolerites, biotite schist, hornblende-schist, chlorite schist, epidiorites, calc granulites, and banded ferruginous-quartzite	Archaeans

The Kaladgi sediments of Saundatti, being the southernmost fringe of the main Kaladgi basin, have a general east-west trend with their dips ranging from 7° to 20° N. The quartzitic sandstones show current bedding, ripple marks, sun cracks and rain prints. They are fine grained to medium grained, being almost made up of quartz (90 to 95 per cent), and can suitably be termed

as orthoquartzites, except for the pebbly quartzitic layer which contains felspars making it subarkosic. Tourmaline, zircon, rutile, chlorite, sericite and hornblende are among the heavy accessories.

The study of heavy mineral mounts and micro-sections of these quartzites reveals the presence of tourmalines with distinct authigenic growths.

Description

The detrital nature of most of the tourmaline grains is clear. Depending on the shape and degree of roundness, they can be termed as very well rounded and angular, according to Krynine and Tuttle. The detrital grains and the new growths are distinct from each other in dichroism exhibiting varying shades of yellow, blue, green, brown, etc. On the basis of dichroism and birefringence, the tourmalines under study have been roughly categorized into the respective members of the isomorphous series as follows:

Heavy mounts

Schorlite	usually black, dark green	purplish brown, greenish brown
Elbaite	pink, pale green, greenish blue	colourless
Dravite	dark yellowish brown	pale yellow
Chromium tourmaline	dark green	pale green, greenish yellow

Micro-sections (0·03 mm.)			*Birefringence*
Schorlite	yellowish brown, black, dark green, blue of various shades	pale to dark yellow, brownish, pale violet, green, yellowish	0·024–0·035
Elbaite	colourless, pink, pale blue	colourless	0·015–0·023
Dravite	pale yellow	colourless	0·022–0·025
Chromium tourmaline	green to bluish green	pale yellow	0·035–0·046

Alty (1933) and others have described the authigenic portion to be colourless and the detrital grain as strongly dichroic. But the overgrowths on the tourmalines described here show distinct dichroism, and in some may even be

considered to be strong. The average birefringence of the host is 0·022 while that of the overgrowth is 0·013 (as determined by Berek's compensator). They are optically negative.

Among the tourmalines, apart from the authigenic development at the termination of the 'c' axis (Alty 1933—Figs. 1, 3 and 7), growths are also observed on the prismatic face of the host (Fig. 6), and in some, the overgrowths completely envelop the host grains (Figs. 2, 4, 5 and 8).

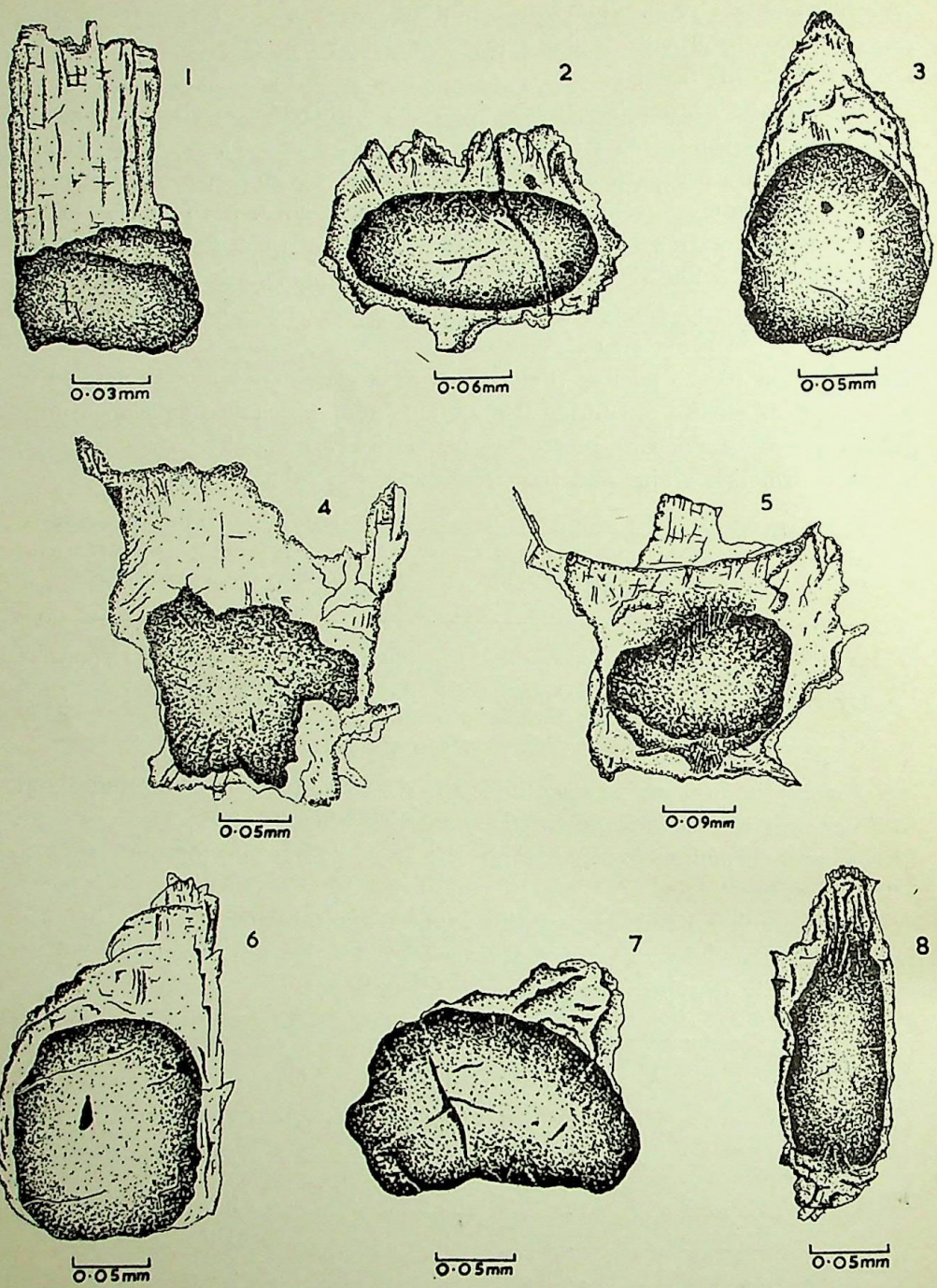
The overgrowths greatly vary in size. A few of them have attained the size of the host while there are some whose development is so much stinted as to appear as mere rims around the detrital grain or as small protuberances. Some others do show an exaggerated growth which may be twice or even thrice the size of the nucleus (Figs. 4, 5). The largest overgrowth noticed so far is that of elbaite measuring 0·300×0·017 mm which has developed on a detrital grain of schorlite measuring 0·150×0·117 mm. The sizes (in mm) of the tourmaline grains and the overgrowths are tabulated below.

	Maximum	Minimum	Average
Host ..	0·8×0·3	0·045×0·03	0·153×0·096
Overgrowth ..	0·3×0·067	0·024×0·003	0·088×0·027

Among the host grains schorlite is the most abundant contributing about 66·2 per cent of the total, followed by dravite which is about 14·3 per cent. Elbaite and chromium tourmaline are the least, accounting for the rest with 9·6 per cent each.

On the other hand, among the overgrowths, elbaite accounts for 61·9 per cent, followed by chromium tourmaline 23·9 per cent, schorlite 9·6 per cent and lastly dravite which is about 4·9 per cent.

	Nuclei percentage	Overgrowths percentage
Schorlite ..	66·2	9·6
Elbaite ..	9·6	61·9
Dravite ..	14·3	4·9
Chromium tourmaline	9·6	23·9



FIGS. 1, 3 and 7. Authigenic development at the termination of 'c' axis. FIG. 6. Authigenic growth on the prismatic face of the host. FIGS. 2, 4, 5, 8. Authigenic growths completely enveloping the host grains. The zone of the roots or reorganization clearly observed in Figs. 5 and 8.

The host grains usually contain inclusions whereas the overgrowths are generally devoid of them and invariably clear. It is not possible to identify the inclusions due to their submicroscopic nature. The overgrowths have a relatively lower relief than the host and are normally found to be angular, with striations and slight fractures. The zone of roots or reorganization is clearly displayed by some of the grains (Figs. 5, 8). In most of the grains perfect optical continuity is observed between the overgrowth and the host, but in a few, slight deviation in optical continuity may be observed as indicated by slight difference in the extinction position of the two. This may be attributed to slight variations in their optical orientation.

Discussion

Alty (1933) reported authigenic tourmalines from the Lower Devonian sediments of Canada and Krynine (1946) from various geological units. In India they have been reported from the Sathyavedu stage (Gundu Rao 1952), the Gulcheru stage of the Cuddapahs (Viswanathiah *et al.* 1963), the Lower Kaladgis of Badami area (Viswanathiah and Govinda Rao Sindhia 1966), from the Lower Kaladgis of Jamkhandi (Govinda Rajulu and Nagaraja 1968), and from the Vindhyan formations of Sone Valley, Mirzapur district (Awasthi 1961).

Various authors in the last few years have drawn attention to some of the factors connected with the authigenic tourmalines. Special mention should be made of Krynine's (1946) excellent work on tourmalines which is highly informative and has proved to be extremely useful in the present study. Based on the colour and morphology of the grains he has recognized five types of major provenances. Making use of his concept, the source for the tourmalines under report can be traced to granitic and pegmatitic provenances. Thus the small idiomorphic tourmalines of dark brown, green or pink colours, with cavities and bubbles are assumed to have been derived from a granitic terrain. Similarly the bigger bluish to greenish blue grains, with fewer inclusions, are suggestive of a pegmatitic parentage.

There is a marked fluctuation in the degree of roundness of the detrital tourmaline grains. Various shapes ranging from angular to well rounded can be observed. Apart from the grains with authigenic growths, there are also prismatic, acicular and euhedral grains which appear to be considerably fresh. This contrast coupled with the different degree of roundness of the grains are the supporting evidences in favour of different provenances.

The authigenic tourmalines speculated to be rare and belonging to the anadiagenetic phase (Fairbridge 1967) have developed new growths by selective absorption of certain chemical constituents from the percolating solutions. They are the probable products of a reaction between the ions of a boron and alumina-rich interstitial water and the primary (detrital) particles, which have acted as nuclei during lithification and diagenesis of the parent rock in

the depositional basin. Each of the three diagenetic phases (syndiagenesis, anadiagenesis and epidiagenesis) are characterized by certain suite of authigenic minerals.

Apart from tourmalines, authigenic growths are also observed in quartz, zircon and sericite. The development of these new growths as a result of diagenesis breeds an interesting study and the process itself is known to be influenced by the nature of the overlying basin water and their fluctuations of oxidation-reduction potential (Eh), hydrogen ion potential (*p*H) and concentration of the concerned ions. The interstitial waters acting as agents of these ions, on saturation, might have led to the precipitation of diagenetic minerals. It is more likely that the minerals and ionic solutions which existed in the fresh Kaladgi sediments were in a state of chemical inequilibrium. To fabricate a state of equilibrium, a chain of events could have been set in motion, whose rates and directions were controlled by the environment. During this resuscitation of equilibrium, selective absorption by ageing gels have led to new minerals and can, therefore, be considered as a vital process in authigenesis.

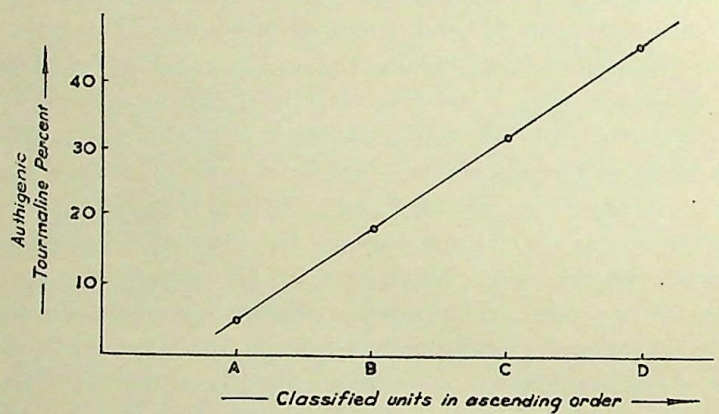


FIG. 9. Graphical representation of the average percentage frequency of authigenic tourmaline for the arenaceous horizons (A, basal conglomerate; B, grey quartzite; C, pink quartzite; and D, pebbly quartzite).

Furthermore, evidences like current bedding, ripple marks, sun cracks, rain prints, etc., found in the parent rock indicate that the authigenic growths have developed under shallow water conditions.

In order to study the horizontal and vertical variations of tourmalines in all the stratigraphic units, 56 representative rock specimens were selected from 14 traverses at regular intervals. Thus each traverse had four specimens representing all the four horizons. Heavy minerals were separated from all these specimens and the tourmaline grains were counted. The average number of tourmaline grains for each traverse works out to be 566 grains, out of which 46 grains show authigenic development and the rest are non-

authigenic. Lateral variations of tourmaline grains are completely neglected as they hardly show any noticeable fluctuations, and each individual unit maintains a uniformity from one end to the other. On the other hand, the vertical variations are well defined when all the four units are taken into consideration, as represented in the graph (Figs. 9 and 10). The average percentage frequency of 46 authigenic and 520 non-authigenic tourmaline grains for each arenaceous horizon are calculated and plotted on the graph accordingly.

In Fig. 9, the representative percentages of authigenic tourmalines in each horizon are calculated for the average 46 grains and represented on the graph. The least number of authigenic growths is in the basal conglomerates amounting to about 4·5 per cent and gradually increases to the maximum in the topmost pebbly quartzitic sandstone beds to about 45·4 per cent. This may also suggest a gradual increase of the suitable chemical ingredients in the percolating waters during deposition, cementation and diagenesis of the

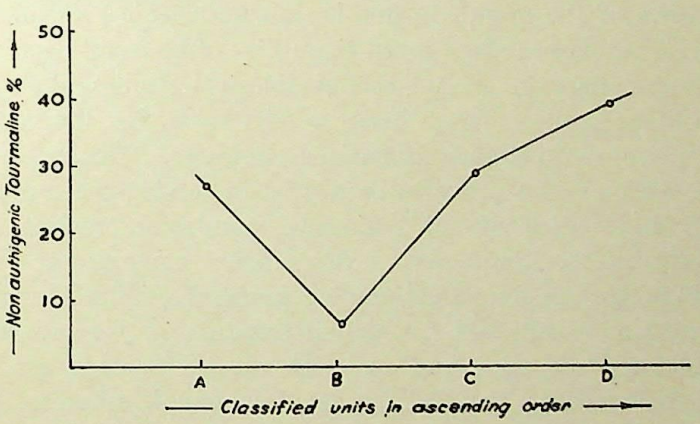


Fig. 10. Graphical representation of the average percentage frequency of non-authigenic tourmaline grains for the arenaceous horizons (A, basal conglomerate; B, grey quartzite; C, pink quartzite; and D, pebbly quartzite).

sediments. This evidence may help to unravel some of the aspects of the palaeogeographical conditions.

In Fig. 10, the percentage distribution of the average 520 grains per traverse of non-authigenic tourmalines is calculated and plotted in a similar manner. The basal conglomeratic bed contains 26·9 per cent of the total number of grains. Here, most of the tourmalines are prismatic, euhedral, pale green or pale blue in colour and sometimes colourless. There is a sudden drop to 5·8 per cent in the immediately overlying pink quartzitic sandstone. From here onwards there is a gradual increase in the successive beds which reaches the maximum of 38·6 per cent in the topmost horizon of pebbly quartzitic sandstone. The tourmalines from the pink quartzite and upwards

are mostly well rounded, dark brown, green or pink in colour. But in the last pebbly horizon there is a mixture of well rounded, angular and also small prismatic euhedral grains.

From these evidences it is surmised that the tourmalines of the basal conglomerate might have been derived from the underlying basement granites and pegmatites. The supply of tourmalines from this source seems to have stopped during the formation of pink quartzites. The nature of the tourmaline grains present here suggests a different percentage, while those that occur in the uppermost horizon of pebbly quartzite appear to have been derived from more than two terrains.

Conclusion

The clear-cut differences in the varieties of tourmalines between the host and the overgrowth, like elbaite on schorlite, dravite on schorlite and the converse, etc., may probably indicate that the detrital grain of tourmaline is not dependent on the identical chemical composition of the complex isomorphous series of the group, to develop an authigenic growth.

The investigation has also proved the utility of both authigenic and non-authigenic tourmalines in certain sedimentological studies like tracing the relationship between the different horizons of a basin, viz. the Kaladgis, and thus may be applied to similar formations elsewhere. Further the concepts discussed above may also prove to be helpful in solving other problems like classification and correlation of sediments, and reconstruction of palaeogeographic conditions. Similarity of depositional environment can also be traced as between the four stratigraphic horizons of the Lower Kaladgis exposed around Saundatti and Jamkhandi respectively (Govinda Rajulu and Nagaraja 1968). It is significant to point out that among the authigenic tourmalines observed in the Kaladgis of Jamkhandi, schorlite and elbaite constitute 60 and 50 per cent respectively among the host grains and the overgrowths. Almost similar data, viz. schorlite (66·2 per cent among the hosts) and elbaite (61·9 per cent among the overgrowths), are obtained in the present investigation where the Lower Kaladgi sandstones of Saundatti are involved.

The development of authigenic minerals in general throws light on the physico-chemical environment which existed in the sediments upon deposition and also helps in defining the physico-chemical boundary limits of diagenesis. We are sure that similar data will be of great use when the tourmalines of the entire Kaladgi basin and also its equivalent formations are studied.

Acknowledgement

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References

Alty, S. W. (1933). Some properties of authigenic tourmaline from the Lower Devonian sediments. *Am. Miner.*, **18**, 351–355.

Awasthi, N. (1961). Authigenic tourmaline and zircon in the Vindhyan formations of Sone Valley, Mirzapur Dist., U.P., India. *J. sedim. Petrol.*, **31**, 482.

Fairbridge, R. W. (1967). Phases of diagenesis and authigenesis. Diagenesis in sediments, developments in sedimentology. Elsvier Publishing Co., Series 8, 19–90.

Gundu Rao, C. (1952). Authigenic tourmaline from Sathyavedu stage (Upper Gondwanas) near Madras. *Curr. Sci.*, **21**, 336.

Govinda Rajulu, B. V., and Nagaraja, H. R. (1968). Occurrence of authigenic tourmalines in the Lower Kaladgi sandstones of Jamkhandi, Mysore State. *J. sedim. Petrol.* (*in press*).

Krynine, P. D. (1946). The tourmaline group in sediments. *J. geol.*, **54**, 65.

Viswanathiah, M. N. (1968). Badami series: A new post-Kaladgi formation of Mysore State. *Bull. geol. Soc. India*, **5**, No. 3, 94–97.

Viswanathiah, M. N., and Govinda Rao Sindhia, M. R. (1966). Authigenic tourmaline from Lower Kaladgi formations (Pre-Cambrians), Badami, Bijapur Dist., Mysore State. *J. Mysore Univ.*, **20**, 46–48.

Viswanathiah, M. N., Govinda Rajulu, B. V., and Raju Philip (1963). Authigenic tourmaline from Gulcheru stage of the Lower Cuddapah formations (Pre-Cambrians), Andhra Pradesh. *Indian Mineralogist*, **4**, 58.

INHOMOGENEOUS MINIMA OF BINARY QUADRATIC, TERNARY CUBIC AND QUATERNARY QUARTIC FORMS IN FIELDS OF FORMAL POWER SERIES*

by SATISH K. AGGARWAL, *Mathematics Department, University of Illinois, Urbana, Illinois*

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In this paper our aim is to prove the analogues of the results of Cassels (1952*a*, 1952*b*) in the field $K\{t\}$ of formal power series in an indeterminate t over the finite field K and equipped with the usual non-Archimedean valuation. Denote by P_n the set of n-tuples of polynomials in t over K. Then we prove the following theorem.

Theorem 1.—Let K be a finite field in which $x^2+m = 0$ is not soluble for some integer m and for which the characteristic is not equal to 2. Let L_1, L_2, L_3, L_4 be four linear forms in variables x, y, z, s with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then we can find elements x_0, y_0, z_0, s_0 in $K\{t\}$ such that

$$\left|L_1^2(X)+mL_2^2(X)\right|\left|L_3^2(X)+mL_4^2(X)\right| \geq \frac{|\Delta|}{e^4}$$

for all $X = (x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$.

Following similar methods we can also prove the following theorems.

Theorem 2.—Let K be a finite field. Let L_1, L_2 be two linear forms in the variables x, y with coefficients in $K\{t\}$ and determinant $\Delta \neq 0$. Then $x_0 \in K\{t\}$ and $y_0 \in K\{t\}$ can be found such that

$$|L_1(X)L_2(X)| \geq \frac{|\Delta|}{e^2}$$

for all $X = (x, y) \equiv (x_0, y_0) \pmod{P_2}$.

Theorem 3.—Let K be a finite field in which x^2+m is irreducible for some integer m and for which the characteristic is not equal to 2. Let L_1, L_2, L_3 be three linear forms in variables x, y, z with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then there exist x_0, y_0, z_0 in $K\{t\}$ such that

$$\left|L_1(X)\right| . \left|L_2^2(X)+mL_3^2(X)\right| \geq \frac{|\Delta|}{e^3}$$

for all $X = (x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$.

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* This is a part of the author's dissertation approved for the Ph.D. degree of Ohio State University in December 1965 written under the guidance of Professor R. P. Bambah.

§ 1. Let K be any field and t be an indeterminate. Let $K[t]$ be the ring of polynomials over K, $K(t)$ the field of rational functions and $K\{t\}$ the field of power series in t^{-1} over K. The elements of $K\{t\}$ are of the form

$$x = \alpha_m t^m + \alpha_{m-1} t^{m-1} + \ldots \text{ (up to } -\infty).$$

In the field $K\{t\}$, we define the usual valuation $|\,|$ as follows:

(i) $|0| = 0$.

(ii) If $x = \alpha_m t^m + \alpha_{m-1} t^{m-1} + \ldots$, $\alpha_m \neq 0$ then $|x| = e^m$.

We also define

$$||x|| = |\alpha_{-1} t^{-1} + \alpha_{-2} t^{-2} + \ldots|.$$

Write $P_n = (K[t])^n = \{X = (x_1, \ldots, x_n)\,;\ x_i \in K[t]\}$ and

$R_n = (K\{t\})^n = \{X = (x_1, \ldots, x_n)\,;\ x_i \in K\{t\}\}$. For $X = (x_1, \ldots, x_n) \in R_n$, we define

$$|X| = \max(|x_1|, \ldots, |x_n|).$$

It is well known that the fields $K\{t\}$, $K(t)$ and the integral domain $K[t]$ are similar in their relative structure to the real field, the rational field and the ring of rational integers respectively. $\|x\|$ corresponds to the fractional part of the real number x, P_n corresponds to n-dimensional fundamental lattice and R_n corresponds to n-dimensional Euclidean space.

§ 2. Let $L_1, L_2, \ldots, L_{r+2s}$ be linear forms in $r+2s = n$ variables such that $L_1, \ldots, L_r$ have real coefficients and $L_{r+1}, \ldots, L_{r+s}$ have complex coefficients and $L_{r+s+1}, \ldots, L_{r+2s}$ are complex conjugates to $L_{r+1}, \ldots, L_{r+s}$ respectively. Let $\Delta \neq 0$ be the determinant of these forms. In the cases $r = 2$, $s = 0$; $r = 1$, $s = 1$ and $r = 0$, $s = 2$, Davenport (1950*a*, 1950*b*, 1951) has proved the existence of positive constants $C_{r,s}$ and reals $\xi_1, \ldots, \xi_n$ such that

$$|L_1 L_2 \ldots L_n| \geq \frac{|\Delta|}{C_{r,s}}$$

for all $(x_1, \ldots, x_n) \equiv (\xi_1, \ldots, \xi_n) \pmod 1$. He proved the theorems with $C_{2,0} = 128$, $C_{1,1} = 8 \times 10^{13}$ and $C_{0,2} = 10^{132}$.

Cassels (1952*a*, 1952*b*) gave an alternative proof with $C_{3,0} = 45{\cdot}2$, $C_{1,1} = 420$ and $C_{0,2} = 5300$. The best values of these C's, however, are not known.

The above-mentioned results have their obvious analogues in connection with the field of formal power series.

Following the techniques of Davenport, Armitage (1957*a*) proved the analogues for finite K in the case of quadratic and cubic forms, but limiting himself only to the cases, where the product $L_1 L_2 \ldots L_n$ does not represent zero non-trivially.

Following the methods of Cassels (1952*a*, 1952*b*) in the real case, the author proved the following theorems in his dissertation.

Theorem 1.—Let K be a finite field. Let L_1, L_2 be two linear forms in variables x, y with coefficients in $K\{t\}$ and determinant $\Delta \neq 0$. Then there exists $(x_0, y_0) \in R_2$ such that

$$|L_1(X) \,.\, L_2(X)| \geq \frac{|\Delta|}{e^2}$$

for all $X = (x, y) \equiv (x_0, y_0) \pmod{P_2}$.

Theorem 2.—Let K be a finite field in which x^2+m is irreducible over K for some integer m and characteristic of K is not equal to 2. Let L_1, L_2, L_3 be three linear forms in variables x, y, z with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then there exists $(x_0, y_0, z_0) \in R_3$ such that

$$\left|L_1(X)\left(L_2^2(X)+mL_3^2(X)\right)\right| \geq \frac{|\Delta|}{e^3}$$

for all $X = (x, y, z) \equiv (x_0, y_0, z_0) \pmod{P_3}$*.

Theorem 3.—Let K be a finite field in which $x^2+m = 0$ is not soluble for some integer m and characteristic of K is not equal to 2. Let L_1, L_2, L_3, L_4 be four linear forms in variables x, y, z, s with coefficients from $K\{t\}$ and determinant $\Delta \neq 0$. Then there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that

$$\left|L_1^2(X)+mL_2^2(X)\right| \,.\, \left|L_3^2(X)+mL_4^2(X)\right| \geq \frac{|\Delta|}{e^4}$$

for all $X = (x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$*.

Remark: The results of all these theorems are best possible.

The three theorems mentioned above may be proved by similar techniques. Since Armitage (1957*a*) proved parts of Theorems 1 and 2, we shall give the details of Theorem 3 only. However, whether we can choose x_0, y_0, z_0, s_0 in $K(t)$, when the product has coefficients in $K(t)$, remains open. It may be remarked that Armitage solved the problem for quadratic and cubic cases, when the product does not represent zero non-trivially.

§ 3. Before proving the actual theorem, we need a few preliminaries.

Since we are given $x^2+m = 0$ is not soluble in K, it is also not soluble in $K\{t\}$. Adjoin α, the root of $x^2+m = 0$ (in the algebraic closure of $K\{t\}$) to the field $K\{t\}$ and we get the field $K\{t\}(\alpha)$ (to be denoted by $K^{(m)}\{t\}$ hereafter). The elements of $K^{(m)}\{t\}$ are of the form $z = x+\alpha y$ where $x, y \in K\{t\}$. We define a function

f: $K^{(m)}\{t\} \to$ the set of non-negative reals by

$$f(z) = f(x+\alpha y) = \max(|x|, |y|)$$

It can be checked that f is a valuation in $K^{(m)}\{t\}$. Since the restriction of f to the field $K\{t\}$ is the original valuation, without ambiguity we shall write

$$f(z) = |z|.$$

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* In my dissertation the theorems are stated for $m = 1$. I am grateful to Professor Bateman for pointing out that the same proof works for general m.

Definition.—Let $z = x+\alpha y \in K^{(m)}\{t\}$. Then we define

$$\|z\| = \max(\|x\|, \|y\|).$$

Definition.—We shall call the elements $z_1 = x_1+\alpha y_1$ and $z_2 = x_1-\alpha y_1$ 'conjugates' of each other and we shall write $z_1 = \tilde{z}_2$ or $z_2 = \tilde{z}_1$

REMARK: If $z_1 = \tilde{z}_2$, then $|z_1| = |z_2|$.

Suppose the given linear forms are

$$L_j(X) = a_jx+b_jy+c_jz+d_js \quad (j = 1, 2, 3, 4).$$

Write

$$L_1' = L_1+\alpha L_2, \quad L_2' = L_1-\alpha L_2,$$

$$L_3' = L_3+\alpha L_4, \quad L_4' = L_3-\alpha L_4, \quad (\alpha^2 = -m).$$

Let Δ' be the determinant of L_1', L_2', L_3', L_4'. It can be checked very easily that $\Delta' = -4m\Delta$. Thus $\Delta' \in K\{t\}$.

Since $x^2+m = 0$ is not soluble in the field K and the characteristic of the field is not 2,

$$|\Delta'| = |-4m\Delta| = |\Delta|.$$

Let M_1, M_2, M_3, M_4 be forms in variables u, v, w, r such that the coefficients of M_j are cofactors of the corresponding coefficients of L_j' $(j = 1, 2, 3, 4)$ respectively. Then

$$L_1'M_1+L_2'M_2+L_3'M_3+L_4'M_4 = \Delta'(xu+yv+zw+sr). \quad \ldots \quad (3.1)$$

It can be checked that $M_1 = \tilde{M}_2$ and $M_3 = \tilde{M}_4$ and $\det(M_1, M_2, M_3, M_4) = \Delta'^3$.

Now we prove a few lemmas.

Lemma 1.—Let $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$ be a finite or infinite sequence from the field $K^{(m)}\{t\}$ such that

$$|\lambda_{n+1}| \geq e|\lambda_n|, \quad \lambda_0 \neq 0.$$

Let $\mu_0, \mu_1, \ldots$ be arbitrary given elements of $K\{t\}$.

Then there exists $\xi \in K^{(m)}\{t\}$ such that

$$\|\lambda_n\xi+\tilde{\lambda}_n\tilde{\xi}+\mu_n\| = e^{-1} \quad (n = 0, 1, 2, \ldots).$$

PROOF: First we note the following. Suppose

$$\lambda_n\xi = \sigma+\alpha\tau.$$

Then

$$\tilde{\lambda}_n\tilde{\xi} = \sigma-\alpha\tau$$

and

$$\lambda_n\xi+\tilde{\lambda}_n\tilde{\xi} = 2\sigma \in K\{t\}.$$

So that for $\lambda_n\xi+\tilde{\lambda}_n\tilde{\xi}$, we have to fix our attention on 'σ' of $\lambda_n\xi$ only.

Without loss of generality, we can assume that the degree of μ_n is at most -1. Suppose

$$\lambda_n = a_{j_n}^{(n)}t^{j_n}+a_{j_{n-1}}^{(n)}t^{j_{n-1}}+\ldots+\alpha\{b_{j_n}^{(n)}t^{j_n}+b_{j_{n-1}}^{(n)}t^{j_{n-1}}+\ldots\}$$

where a's and b's belong to K and at least one of $a_{j_n}^{(n)}$ and $b_{j_n}^{(n)}$ is not zero. Let

$$\mu_n = r_{-1}^{(n)}t^{-1}+r_{-2}^{(n)}t^{-2}+\dots \quad .$$

Take

$$\xi = \alpha_{-j_0-1}t^{-j_0-1}+\alpha_{-j_0-2}t^{-j_0-2}+\dots+\alpha\{\beta_{-j_0-1}t^{-j_0-1}+\beta_{-j_0-2}t^{-j_0-2}+\dots\}$$

where α's and β's belong to K. We shall prove that with suitable choice of α's and β's, we have the required result. We choose these by induction. Coefficient of t^{-1} in $\lambda_0\xi+\tilde{\lambda}_0\tilde{\xi}+\mu_0 = 2\{a_{j_0}^{(0)}\alpha_{-j_0-1}-mb_{j_0}^{(0)}\beta_{-j_0-1}\}+r_{-1}^{(0)}$.

Since at least one of $a_{j_0}^{(0)}$, $b_{j_0}^{(0)}$ is not zero, we can choose α_{-j_0-1}, $\beta_{-j_0-1} \in K$ such that coefficient of t^{-1} in $\lambda_0\xi+\tilde{\lambda}_0\tilde{\xi}+\mu_0$ is not zero. Suppose we have selected already

$$\alpha_{-(j_{0+1})}, \dots, \alpha_{-(j_{k-1}+1)}, \beta_{-(j_{0+1})}, \dots, \beta_{-(j_{k-1}+1)}.$$

Since $j_k \geq j_{k-1}+1$, choose $\alpha_{-(j_{k-1}+2)}, \dots, \alpha_{-j_k}$, $\beta_{-(j_{k-1}+2)}, \dots, \beta_{-j_k}$, arbitrarily and $\alpha_{-(j_k+1)}$, $\beta_{-(j_k+1)}$ such that t^{-1} has a non-zero coefficient in

$$\lambda_k\xi+\tilde{\lambda}_k\tilde{\xi}+\mu_k$$

$$= 2\left\{\left(\alpha_{-(j_k+1)}a_{j_k}^{(k)}+\dots+\alpha_{-(j_0+1)}a_{j_0}^{(k)}\right)-m\left(\beta_{-(j_k+1)}b_{j_k}^{(k)}+\dots+\beta_{-(j_0+1)}b_{j_0}^{(k)}\right)\right\}+r_{-1}^{(k)}.$$

This is possible, since $(a_{j_k}^{(k)}, b_{j_k}^{(k)}) \neq (0, 0)$.

Thus the induction is complete and lemma is proved.

In the subsequent discussion K stands for a finite field.

Lemma 2.—Let $L_1(U), \dots, L_n(U)$ be n linear forms over $K\{t\}$ in n variables $u_1, \dots, u_n$ with determinant $\Delta \neq 0$. Then for every constant $C > 0$, there exists only a finite number of elements U of P_n satisfying

$$\max(|L_1(U)|, \dots, |L_n(U)|) \leq C.$$

Proof: We define a new function

$$F(U) = \max(|L_1(U)|, \dots, |L_n(U)|)$$

and by a result of Mahler (1941), we get

$$F(U) \geq \gamma|U|$$

for a constant $\gamma > 0$ and for all $U \in R_n$ and our lemma follows immediately from this.

Since K is a finite field, $K\{t\}$ is locally compact and we have

Lemma 3.—Every infinite bounded sequence in $K\{t\}$ (in the sense of valuation) has a convergent subsequence.

Lemma 4.—Let p, q be integers such that

$$e^{2p+2q} \geq e^{-3}|\Delta|^3.$$

Then there exists a non-zero vector $U = (u, v, w, r) \in P_4$ such that

$$|M_1(U)| = |M_2(U)| \leq e^p, |M_3(U)| = |M_4(U)| \leq e^q \qquad .. \quad (3.2)$$

Further the inequalities in (3.2) have only a finite number of solutions from P_4.

PROOF: Since $M_1 = \tilde{M}_2$ and $M_3 = \tilde{M}_4$, let

$$M_1 = N_1+\alpha N_2,\ M_2 = N_1-\alpha N_2$$

$$M_3 = N_3+\alpha N_4,\ M_4 = N_3-\alpha N_4$$

where N_1, N_2, N_3, N_4 are linear forms with coefficients from $K\{t\}$. Then we can see

$$0 \neq |\det(M_1, M_2, M_3, M_4)| = |\det(N_1, N_2, N_3, N_4)| = |\Delta|^3.$$

Since $e^{2p+2q} \geq e^{-3}|\Delta|^3$ by a theorem of Mahler (*see* Armitage 1957*b*), there exists a non-zero $U = (u, v, w, r) \in P_4$ satisfying

$$|N_1(U)| \leq e^p,\ |N_2(U)| \leq e^p,\ |N_3(U)| \leq e^q,\ |N_4(U)| \leq e^q \quad .. \quad (3.3)$$

But, by definition, for any $U \in P_4$,

$$|M_1(U)| = |M_2(U)| = \max(|N_1(U)|, |N_2(U)|)$$

and

$$|M_3(U)| = |M_4(U)| = \max(|N_3(U)|, |N_4(U)|).$$

So non-zero $U \in P_4$, which satisfies (3.3), must also satisfy (3.2) and this proves the first part.

To prove that (3.2) has only a finite number of solutions $U \in P_4$, we first observe that any solution of (3.2) is also a solution of (3.3). But, by Lemma 2, (3.3) has only a finite number of solutions $U = (u, v, w, r) \in P_4$. This proves the lemma.

REMARK: Let K be a field (not necessarily finite), in which $x^2+m = 0$ is not soluble. Let L_1, L_2, L_3, L_4 be four linear forms in variables $(x, y, z, s) = X$ with determinant $\Delta \neq 0$. Then Lemma 4 incidentally proves that there exists a non-zero vector $X \in P_4$, satisfying

$$|L_1^2(X)+mL_2^2(X)|\,|L_3^2(X)+mL_4^2(X)| \leq e^{-3}|\Delta|.$$

Now we distinguish the following three cases:

(i) $M_1 \neq 0 \neq M_2$, $M_3 \neq 0 \neq M_4$ for any non-zero $U \in P_4$.

(ii) One and only one of the pairs (M_1, M_2), (M_3, M_4), say (M_3, M_4) assumes the value $(0, 0)$ for some non-zero vector $U \in P_4$.

(iii) $M_1 = 0 = M_2$ for some non-zero $U \in P_4$ and $M_3 = 0 = M_4$ for another non-zero vector $U \in P_4$.

§ 4. *Proof of Theorem 3 in Case I*

First we shall prove a few more lemmas.

Lemma 5.—Let $M \geq 1$ be any given integer. Then there exists a finite sequence of values α_n, $\beta_n = \tilde{\alpha}_n$, γ_n, $\delta_n = \tilde{\gamma}_n (0 \leq n \leq N)$ of M_1, M_2, M_3, M_4

respectively corresponding to a sequence $\mathfrak{I}_M$ of elements $(u_n, v_n, w_n, r_n) \in P_4$ with gcd $(u_n, v_n, w_n, r_n) = 1$ and satisfying the following properties:

(i) $|\alpha_n^2 \gamma_n^2| \leq e^{-2} |\Delta|^3$.

(ii) $|\alpha_n| \leq e^{-1} |\alpha_{n-1}|$, $(n \geq 1)$.

(iii) $|\alpha_{n-1}^2 \gamma_n^2| \leq |\Delta|^3$, $(n \geq 1)$.

(iv) $|\gamma_0| \leq e^{-M}$, $|\alpha_0| \geq e^M$.

(v) $|\alpha_N| \leq e^{-M}$.

(vi) $|\gamma_n| \geq |\gamma_{n-1}|$, $(n \geq 1)$.

PROOF: Let $|\Delta| = e^d$. Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r)| &\leq \max\left(e^{M-1},\ e^{M+\left[\frac{3d}{2}\right]-2}\right) \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| &\leq e^{-M+1} \end{aligned}\right\} \quad .. \quad (4.1)$$

By Lemma 4, the above inequalities have a non-zero solution $U = (u, v, w, r) \in P_4$ and the number of such solutions is finite. Consider

$$\min |M_3(u, v, w, r)|$$

as (u, v, w, r) runs over non-zero solutions from P_4 of inequalities (4.1). Since the number of these solutions is finite, this minimum is attained and by hypothesis, it is not zero. Let this minimum be e^{-T}. Then $T \geq M-1$. Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r)| &\leq e^{T+\left[\frac{3d}{2}\right]} \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| &\leq e^{-T-1} \end{aligned}\right\} \quad .. \quad .. \quad (4.2)$$

By Lemma 4, the inequalities (4.2) have at least one non-zero solution $U = (u, v, w, r) \in P_4$. Further the number of such solutions is finite. Hence min $|M_3(u, v, w, r)|$ taken over non-zero vectors in P_4 satisfying (4.2) is attained, say at (u_0, v_0, w_0, r_0). Obviously, the gcd $(u_0, v_0, w_0, r_0) = 1$.

Let $\alpha_0 = M_1(u_0, v_0, w_0, r_0)$, $\beta_0 = M_2(u_0, v_0, w_0, r_0)$,

$\gamma_0 = M_3(u_0, v_0, w_0, r_0)$ and $\delta_0 = M_4(u_0, v_0, w_0, r_0)$. Then

$|\gamma_0| = |\delta_0| \leq e^{-T-1} \leq e^{-M}$ (since $T \geq M-1$).

Also for every non-zero $U = (u, v, w, r) \in P_4$ satisfying (4.2), we must have $|M_1(U)| > e^{M-1}$. Otherwise, we have $|M_1(U)| \leq e^{M-1}$ and then this U satisfies (4.1), which contradicts the minimal nature of T. Hence, in particular $|\alpha_0| \geq e^M$.

Since (u_0, v_0, w_0, r_0) satisfies (4.2), (u_0, v_0, w_0, r_0) also satisfies condition (i) of Lemma 5. Suppose there exists non-zero $U = (u, v, w, r) \in P_4$ satisfying $|M_1(U)| \leq e^{-1} |\alpha_0|$. Then we claim $|M_3(U)| \geq |\gamma_0|$. For suppose we have $|M_3(U)| < |\gamma_0|$. Then this non-zero U also satisfies (4.2), and this contradicts the minimal nature of $|\gamma_0|$.

Now we shall construct by induction a sequence $\mathfrak{I}_M$ satisfying the conditions of Lemma 5. (u_0, v_0, w_0, r_0) satisfies the conditions of the lemma. Suppose $(u_0, v_0, w_0, r_0), \ldots, (u_{n-1}, v_{n-1}, w_{n-1}, r_{n-1})$ have been constructed. Let

$$\alpha_i = M_1(u_i, v_i, w_i, r_i),\ \beta_i = M_2(u_i, v_i, w_i, r_i)$$

$$\gamma_i = M_3(u_i, v_i, w_i, r_i) \text{ and } \delta_i = M_4(u_i, v_i, w_i, r_i) \text{ for all } i.$$

Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| = |M_2(u, v, w, r)| &\leq e^{-1}|\alpha_{n-1}|, \\ |M_3(u, v, w, r)| = |M_4(u, v, w, r)| &\leq |\alpha_{n-1}|^{-1}e^{-\left[\frac{3d}{2}\right]}. \end{aligned}\right\} \quad \ldots \quad (4.3)$$

By Lemma 4, there exists at least one non-zero solution $U = (u, v, w, r) \in P_4$ satisfying (4.3) and there are only a finite number of such solutions. As before, min $|M_3(U)|$, where minimum is being taken over all non-zero $U \in P_4$ satisfying (4.3) is attained, say at (u_n, v_n, w_n, r_n). Conditions (i), (ii), (iii) of the lemma are obviously satisfied. It must satisfy also condition (vi), for otherwise, it would have been selected at the previous stage. Since $|\alpha_n| \leq e^{-1}|\alpha_{n-1}|$ for all n, we get integer N' such that

$$|\alpha_n| \leq e^{-M}$$

for all $n \geq N'$. We stop at such an N'.

REMARK: 1. Clearly $\mathfrak{I}_M$ is not unique.

2. If $M_1 > M_2$, every $\mathfrak{I}_{M_1}$ is an $\mathfrak{I}_{M_2}$ also.

Lemma 6.—There exists

$$(x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) \in R_4,$$

such that

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1}$$

for all $(x, y, z, s) \equiv (x_0^{(M)},\ y_0^{(M)},\ z_0^{(M)},\ s_0^{(M)}) \pmod{P_4}$

and for all $n (0 \leq n \leq N)$.

PROOF: Consider

$$\lambda_n = \frac{\alpha_{N-n}}{\Delta'},\quad \mu_n = 0 \quad (n = 0, 1, \ldots, N).$$

Owing to condition (ii) of Lemma 5,

$$|\lambda_n| \geq e|\lambda_{n-1}| \quad (n = 1, \ldots, N).$$

We can, therefore, apply Lemma 1 and get $\xi \in K^{(m)}\{t\}$, such that

$$\left|\left|\frac{\alpha_n\xi + \tilde{\alpha}_n\tilde{\xi}}{\Delta'}\right|\right| = \left|\left|\frac{\alpha_n\xi + \beta_n\tilde{\xi}}{\Delta'}\right|\right| = e^{-1} (n = 0, 1, \ldots, N).$$

Let $\xi = \eta + \alpha\zeta$; $\eta, \zeta \in K\{t\}$. Define

$(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \in R_4$ by the equations

$$L_1(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = \eta, \quad L_2(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = \zeta,$$

$$L_3(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = 0 = L_4(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}).$$

Then $(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)})$ also satisfies

$$L_1'(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = \xi, \quad L_2'(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = \tilde{\xi}$$

$$L_3'(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) = 0 = L_4'(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}).$$

Substituting $x = x_0^{(M)}$, $y = y_0^{(M)}$, $z = z_0^{(M)}$, $s = s_0^{(M)}$, $u = u_n$, $v = v_n$, $w = w_n$ and $r = r_n$ in (3.1), we get

$$x_0^{(M)}u_n + y_0^{(M)}v_n + z_0^{(M)}w_n + s_0^{(M)}r_n = \frac{\xi\alpha_n + \tilde{\xi}\tilde{\alpha}_n}{\Delta'}.$$

Hence

$$\left\| x_0^{(M)}u_n + y_0^{(M)}v_n + z_0^{(M)}w_n + s_0^{(M)}r_n \right\|$$

$$= \left\| \frac{\xi\alpha_n + \tilde{\xi}\tilde{\alpha}_n}{\Delta'} \right\| = e^{-1} (0 \leq n \leq N).$$

Further if $(x, y, z, s) \equiv (x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \pmod{P_4}$, we have

$$(xu_n + yv_n + zw_n + sr_n) - (x_0^{(M)}u_n + y_0^{(M)}v_n + z_0^{(M)}w_n + s_0^{(M)}r_n)$$

$$= (x - x_0^{(M)})u_n + (y - y_0^{(M)})v_n + (z - z_0^{(M)})w_n + (s - s_0^{(M)})r_n$$

and the right hand side is an element of $K[t]$. Consequently

$$\| xu_n + yv_n + zw_n + sr_n \| = \left\| x_0^{(M)}u_n + y_0^{(M)}v_n + z_0^{(M)}w_n + s_0^{(M)}r_n \right\| = e^{-1} (0 \leq n \leq N).$$

Hence

$$|xu_n + yv_n + zw_n + sr_n| \geq \| xu_n + yv_n + zw_n + sr_n \| = e^{-1} (0 \leq n \leq N).$$

This proves the lemma.

Lemma 7.—There exists a doubly infinite sequence of polynomials (u_n, v_n, w_n, r_n) $(-\infty < n < \infty)$ with gcd $(u_n, v_n, w_n, r_n) = 1$, such that conditions (i), (ii), (iii) and (vi) of Lemma 5 hold and

$$\lim_{n \to \infty} |M_1(u_n, v_n, w_n, r_n)| = 0 = \lim_{n \to -\infty} |M_3(u_n, v_n, w_n, r_n)|.$$

Also

$$\lim_{n \to \infty} |M_3(u_n, v_n, w_n, r_n)| = \infty = \lim_{n \to -\infty} |M_1(u_n, v_n, w_n, r_n)|.$$

Also there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that for all $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$, the condition

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1} (-\infty < n < \infty)$$

is satisfied.

PROOF: Let L be any given integer. First we assert that there are only a finite number of pairs of non-zero elements $(u, v, w, r), (u', v', w', r') \in P_4$ satisfying

(a) $|M_1(u, v, w, r)|^2 |M_3(u', v', w', r')|^2 \leq |\Delta|^3$,

(b) $|M_1(u, v, w, r)|^2 |M_3(u, v, w, r)|^2 \leq e^{-2} |\Delta|^3$,

(c) $|M_1(u', v', w', r')| \leq e^L \leq e^{-1} |M_1(u, v, w, r)|$.

From (a) and (c) we get

$$|M_3(u', v', w', r')|^2 \leq e^{-2L-2} |\Delta|^3.$$

Therefore

$$|M_3(u', v', w', r')| \leq e^{-L-1+[3d/2]}.$$

But for $(u', v', w', r') \in P_4$,

$$|M_1(u', v', w', r')| = |M_2(u', v', w', r')|$$

and

$$|M_3(u', v', w', r')| = |M_4(u', v', w', r')|.$$

So we have

$$|M_1(u', v', w', r')| = |M_2(u', v', w', r')| \leq e^L \text{ [from (c)]}$$

and

$$|M_3(u', v', w', r')| = |M_4(u', v', w', r')| \leq e^{-L-1+[3d/2]}.$$

By Lemma 4, there exist only a finite number of $(u', v', w', r') \in P_4$ satisfying the above.

Further given non-zero $(u', v', w', r') \in P_4$, we assert there are only a finite number of non-zero $(u, v, w, r) \in P_4$. Since by assumption, $M_3(u', v', w', r') \neq 0$, from (a) we get

$$|M_1(u, v, w, r)|^2 \leq |\Delta|^3 \cdot |M_3(u', v', w', r')|^{-2}.$$

From (b) we get,

$$\begin{aligned} |M_3(u, v, w, r)|^2 &\leq e^{-2} |\Delta|^3 |M_1(u, v, w, r)|^{-2} \\ &\leq e^{-4-2L} |\Delta|^3. \end{aligned}$$

Now by using Lemma 2, we can see that there are only a finite number of $(u, v, w, r) \in P_4$ satisfying these.

Now suppose $M \geq 1$ is a given integer. Construct a sequence $\mathfrak{I}_M$ satisfying the conditions of Lemma 5. By conditions (iv), (v) and (ii) of Lemma 5, there exists a unique integer N_1 such that $N \geq N_1 \geq 1$ and

$$|\alpha_{N_1}| \leq 1 \leq e^{-1} |\alpha_{N_1-1}|.$$

Let $\mathfrak{I}'_M$ be the sequence of (u'_n, v'_n, w'_n, r'_n) defined by

$$(u'_n, v'_n, w'_n, r'_n) = (u_{n+N_1}, v_{n+N_1}, w_{n+N_1}, r_{n+N_1}) \text{ for}$$
$$-N_1 \leq n \leq N_2 = N - N_1.$$

Let $\alpha_n', \beta_n', \gamma_n', \delta_n'$ be the values assumed by M_1, M_2, M_3, M_4 respectively at (u_n', v_n', w_n', r_n'). Then conditions (i), (ii), (iii) and (vi) of Lemma 6 hold for

$$(u_n', v_n', w_n', r_n') \text{ and}$$

$$|\alpha_0'| \leq 1 \leq e^{-1}|\alpha_{-1}'|, \; |\alpha_{N_2}'| \leq e^{-M}, \; |\alpha_{-N_1}'| \geq e^M \quad .. \qquad .. \quad (4.4)$$

We now construct $\mathfrak{F}$ by a diagonal process. By the observation already made, there are only a finite number of possibilities for

$$(u_0', v_0', w_0', r_0'), \quad (u_{-1}', v_{-1}', w_{-1}', r_{-1}')$$

and so at least one pair of these must occur infinitely often, as M runs through the set of natural numbers. We fix one such possible pair, say $(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0)$, $(\bar{u}_{-1}, \bar{v}_{-1}, \bar{w}_{-1}, \bar{r}_{-1})$. We now consider only those $\mathfrak{F}_M'$'s which contain these.

For these M, conditions (4.4) imply that for all M large enough $N_2 \geq 1$, for otherwise $M_1(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0) = 0$, which contradicts the hypothesis.

Denote by $\bar{\alpha}_n, \bar{\beta}_n, \bar{\gamma}_n, \bar{\delta}_n$ the values assumed by M_1, M_2, M_3, M_4 respectively at $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$. Now

$$|M_1(u_1', v_1', w_1', r_1')| \leq e^{-1}|\bar{\alpha}_0| \leq e^{-1}|M_1(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0)|$$

for each $\mathfrak{F}_M'$.

Again, there are only a finite number of possible choices for $(u_1', v_1', w_1', r_1') \in P_4$. We pick out $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1) \in P_4$ which occurs infinitely often. Let $\mathfrak{D}_2$ be the infinite subsequence of $\mathfrak{D}_1$, consisting of those $\mathfrak{F}_M'$'s to which $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)$ belong. We now consider only these $\mathfrak{F}_M'$'s. For these $\mathfrak{F}_M'$'s, if M is large enough, $N_1 \geq 2$.

Consider

$$|M_1(\bar{u}_{-1}, \bar{v}_{-1}, \bar{w}_{-1}, \bar{r}_{-1})| \leq |\bar{\alpha}_{-1}| \leq e^{-1}|M_1(u_{-2}', v_{-2}', w_{-2}', r_{-2}')|$$

for all those $\mathfrak{F}_M'$'s. As before, there are only a finite number of choices for $(u_{-2}', v_{-2}', w_{-2}', r_{-2}') \in P_4$. We pick out $(\bar{u}_{-2}, \bar{v}_{-2}, \bar{w}_{-2}, \bar{r}_{-2})$, which occurs infinitely often. Let $\mathfrak{D}_3$ be the infinite subsequence of $\mathfrak{D}_2$ composed of those $\mathfrak{F}_M'$'s to which $(\bar{u}_{-2}, \bar{v}_{-2}, \bar{w}_{-2}, \bar{r}_{-2})$ belongs and we consider only these $\mathfrak{F}_M'$'s. For these $\mathfrak{F}_M'$'s, $N_2 \geq 2$ if M is large enough.

Continuing this process, we get a doubly infinite sequence $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$ $(-\infty < n < \infty)$, which satisfies conditions (i), (ii), (iii) and (vi) of Lemma 5.

Obviously, by construction

$$\lim_{n \to \infty} |\bar{\alpha}_n| = 0, \quad \lim_{n \to -\infty} |\bar{\alpha}_n| = \infty.$$

Due to condition (i) of this lemma, we must have

$$\lim_{n \to -\infty} |\bar{\gamma}_n| = 0.$$

Also $\lim_{n\to\infty} |\bar{\gamma}_n| = \infty$, for otherwise, there exists an integer k such that

$$|\bar{\gamma}_n| \leq e^k$$

for all n. Then

$$|M_1(u, v, w, r)| = |M_2(u, v, w, r)| \leq 1$$

$$|M_3(u, v, w, r)| = |M_4(u, v, w, r)| \leq e^k$$

has infinitely many solutions $(u, v, w, r) \in P_4$, which by using Lemma 2, we can see, is not true. Thus this sequence has all the required properties.

Since there is no danger of confusion, we now suppress the bar on (u_n, v_n, w_n, r_n) and $\alpha_n, \beta_n, \gamma_n, \delta_n$.

During our construction of double sequence, we have used the subsequences $\mathfrak{D}_1, \mathfrak{D}_2, \ldots$ of the sequences $\mathfrak{T}_M'$'s which have the properties

(i) $\mathfrak{D}_{i+1}$ is a subsequence of $\mathfrak{D}_i$

(ii) Given any finite set of (u_j, v_j, w_j, r_j)'s, this is a subset of $\mathfrak{D}_j$ for j large.

Let $\mathfrak{T}_j^*$ be the first member of $\mathfrak{D}_j$. Then the sequences $\mathfrak{T}_j^*$ has the following properties:

(1) $\mathfrak{T}_j^* = \mathfrak{T}_{m_j}$ for some m_j; $m_j \to \infty$ as $j \to \infty$.

(2) Each finite set of (u_j, v_j, w_j, r_j) is a subset of $\mathfrak{T}_j^*$ for all j large enough.

By Lemma 6, there exists a sequence

$$\{(x_0^{(m_j)}, \ y_0^{(m_j)}, \ z_0^{(m_j)}, \ s_0^{(m_j)}); \ j = 1, 2, \ldots\}$$

and we may assume that

$$|x_0^{(m_j)}| < 1, \quad |y_0^{(m_j)}| < 1, \quad |z_0^{(m_j)}| < 1, \quad |s_0^{(m_j)}| < 1$$

for all j.

By using Lemma 3, we can see that

$$\{(x_0^{(m_j)}, \ y_0^{(m_j)}, \ z_0^{(m_j)}, \ s_0^{(m_j)}), j = 1, 2, \ldots\}$$

has a convergent subsequence tending to a limit point (x_0, y_0, z_0, s_0) (say). Without loss of generality, assume this itself is convergent. This choice of (x_0, y_0, z_0, s_0) does what is required. For take $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$. Let (u_n, v_n, w_n, r_n) be any member of this double sequence. For all j large enough, $(u_n, v_n, w_n, r_n) \in \mathfrak{T}_j^* = \mathfrak{T}_k$ (say),

and

$$\begin{aligned}|xu_n+yv_n+zw_n+sr_n| &= \big|(x-x_0+x_0^{(k)})u_n+(y-y_0+y_0^{(k)})v_n+(z-z_0+z_0^{(k)})w_n+(s-s_0+s_0^{(k)})r_n+\\ &\quad (x_0-x_0^{(k)})u_n+(y_0-y_0^{(k)})v_n+(z_0-z_0^{(k)})w_n+(s_0-s_0^{(k)})r_n\big|.\end{aligned}$$

Since

$$(x-x_0+x_0^{(k)},\ y-y_0+y_0^{(k)},\ z-z_0+z_0^{(k)},\ s-s_0+s_0^{(k)}) \equiv (x_0^{(k)},\ y_0^{(k)},\ z_0^{(k)},\ s_0^{(k)}) \pmod{P_4},$$

$$|(x-x_0+x_0^{(k)})u_n+(y-y_0+y_0^{(k)})v_n+(z-z_0+z_0^{(k)})w_n+(s-s_0+s_0^{(k)})r_n| \geq e^{-1} \text{ for all } n.$$

If $j \to \infty$, $k \to \infty$; therefore, if j is large enough

$$|(x_0-x_0^{(k)})u_n+(y_0-y_0^{(k)})v_n+(z_0-z_0^{(k)})w_n+(s_0-s_0^{(k)})r_n| < e^{-1}$$

and hence

$$|xu_n+yv_n+zw_n+sr_n| \geq e^{-1} \text{ for all } n.$$

This completes the proof of Lemma 7.

Proof of Theorem 3 in Case I.—Let (x_0, y_0, z_0, s_0) and the doubly infinite sequence (u_n, v_n, w_n, r_n) $(-\infty < n < \infty)$ be given by Lemma 7. Take any $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$. Suppose $a, b = \tilde{a}, c, d = \tilde{c}$ are the values assumed by L_1', L_2', L_3', L_4' respectively at (x, y, z, s).

First we assert neither of a, b, c, d is zero. For suppose $a = 0 = b$. Then from (3.1) we get for any n,

$$xu_n+yv_n+zw_n+sr_n = \frac{c\gamma_n+d\delta_n}{\Delta'}.$$

So

$$|xu_n+yv_n+zw_n+sr_n| = \left|\frac{c\gamma_n+d\delta_n}{\Delta'}\right| \leq \left|\frac{c\gamma_n}{\Delta}\right|.$$

(Since $|c\gamma_n| = |d\delta_n|$ and $|\Delta'| = |\Delta|$.) Making $n \to -\infty$, we get

$$|xu_n+yv_n+zw_n+sr_n| \to 0$$

and this contradicts the fact that $|xu_n+yv_n+zw_n+sr_n| \geq e^{-1}$ for all n. Hence $a \neq 0 \neq b$. Similarly $c \neq 0 \neq d$.

Since $|\gamma_n| \to \infty$ as $n \to \infty$ and $|\gamma_n| \to 0$ as $n \to -\infty$ and $|\gamma_{n+1}| \geq |\gamma_n|$, we can choose an integer n
such that

$$|\gamma_n|^2 \leq |\Delta|^{\frac{1}{2}}\left|\frac{a}{c}\right| \leq |\gamma_{n+1}|^2. \quad .. \quad .. \quad .. \quad (4.5)$$

From condition (iii) of Lemma 7, we get

$$|\alpha_n|^2 \leq |\Delta|^3|\gamma_{n+1}|^{-2} \leq \left|\frac{c}{a}\right||\Delta|^{\frac{1}{2}} \quad .. \quad .. \quad .. \quad (4.6)$$

Put

$$p = |ab||\alpha_n\beta_n| = |a|^2|\alpha_n|^2, \quad q = |cd||\gamma_n\delta_n| = |c|^2|\gamma_n|^2$$

and

$$k^2 = |abcd||\Delta|^3 = |ac|^2|\Delta|^3.$$

Then from (4.5) and (4.6), we get

$$p = |a|^2||\alpha_n|^2 \leq |ac||\Delta|^{\frac{1}{2}} = k,$$
$$q = |c|^2|\gamma_n|^2 \leq |ac||\,|\Delta|^{\frac{1}{2}} = k.$$

From (3.1), we get

$$|\Delta'(xu_n+yv_n+zw_n+sr_n)| = |a\alpha_n+b\beta_n+c\gamma_n+d\delta_n| \leq \max(|a\alpha_n|, |c\gamma_n|)$$

i.e. $$|\Delta(xu_n+yv_n+zw_n+sr_n)|^2 \leq \max(|a\alpha_n|^2, |c\gamma_n|^2) \leq k.$$

But from Lemma 7, we get

$$|xu_n+yv_n+zw_n+sr_n)| \geq e^{-1}$$

$$|\Delta(xu_n+yv_n+zw_n+sr_n)|^2 \geq e^{-2}|\Delta|^2.$$

Hence

$$k \geq e^{-2}|\Delta|^2$$

or $$|abcd|^{\frac{1}{2}}|\Delta|^{\frac{3}{2}} \geq e^{-2}|\Delta|^2$$

or $$|abcd| \geq |\Delta|/e^4.$$

This proves the theorem in this case.

§ 5. *Proof of Theorem 3 in Case II*

Here we prove the following lemmas.

Lemma 5'.—*Let* $M > 0$ be a given integer. Then there exists a finite sequence of values $\alpha_n, \beta_n = \tilde{\alpha}_n, \gamma_n, \delta_n = \tilde{\gamma}_n$ $(0 \leq n \leq N)$ of M_1, M_2, M_3, M_4 respectively corresponding to a sequence $\mathfrak{S}_M$ of polynomials (u_n, v_n, w_n, r_n) $(0 \leq n \leq N)$ with gcd $(u_n, v_n, w_n, r_n) = 1$ satisfying conditions (i), (ii), (iii), (v) and (vi) of Lemma 5 and

(iv'): $$\gamma_0 = 0, \gamma_n \not\equiv 0 \quad \text{for} \quad 0 < n \leq N.$$

Proof: By hypothesis there exists $(\alpha, \beta, \gamma, \delta) \in P_4$, $(\alpha, \beta, \gamma, \delta) \not\equiv 0$ and with gcd $(\alpha, \beta, \gamma, \delta) = 1$, such that

$$M_3(\alpha, \beta, \gamma, \delta) = 0 = M_4(\alpha, \beta, \gamma, \delta).$$

Consider the solutions of

$$\left.\begin{aligned} |M_1(u, v, w, r)| &= |M_2(u, v, w, r| \leq |M_1(\alpha, \beta, \gamma, \delta)| \\ |M_3(u, v, w, r)| &= |M_4(u, v, w, r)| = 0 \leq 1. \end{aligned}\right\} \quad .. \quad (5.1)$$

By hypothesis, (5.1) has at least one non-zero solution $(\alpha, \beta, \gamma, \delta) \in P_4$. Using Lemma 2, we can see that (5.1) has only a finite number of solutions $U = (u, v, w, r) \in P_4$. Consider

$$\min |M_1(u, v, w, r)|,$$

the minimum being taken over all non-zero $(u, v, w, r) \in P_4$ with gcd $(r, v, w, r) = 1$ satisfying (5.1). As there are only a finite number of solutions, minimum is attained, say at (u_0, v_0, w_0, r_0). Then $M_1(u_0, v_0, w_0, r_0) \not\equiv 0$.

Now suppose there exists non-zero $(u, v, w, r) \in P_4$ with gcd $(u, v, w, r) = 1$, such that

$$|M_1(u, v, w, r)| \leq e^{-1}|M_1(u_0, v_0, w_0, r_0)|.$$

Then $M_3(u, v, w, r) \neq 0$, for otherwise, this (u, v, w, r) contradicts the choice of (u_0, v_0, w_0, r_0). In particular, if there exists (u_1, v_1, w_1, r_1) satisfying condition (ii) of Lemma 5, then $\gamma_1 \neq 0$.

With this choice of (u_0, v_0, w_0, r_0), we proceed as in Lemma 5 and determine the sequence $\mathfrak{I}_M$ satisfying conditions (i), (ii), (iii), (v) and (vi) of Lemma 5.

Since $|\gamma_n| \geq |\gamma_{n-1}|$ and $\gamma_1 \neq 0$, we get condition (iv′) of Lemma 5.

Lemma 6′.—For the sequence obtained in Lemma 5′, there exists

$$(x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \in R_4$$

such that

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1}$$

for all

$$(x, y, z, s) \equiv (x_0^{(M)}, y_0^{(M)}, z_0^{(M)}, s_0^{(M)}) \pmod{P_4}$$

and for all n $(0 \leq n \leq N)$.

The proof of Lemma 6′ is the same as that of Lemma 6.

Lemma 7′.—There exists an infinite sequence $\mathfrak{I}$ of

$$(u_n, v_n, w_n, r_n) \in P_4 \ (0 \leq n < \infty) \text{ with } \gcd(u_n, v_n, w_n, r_n) = 1$$

satisfying conditions (i), (ii), (iii) and (vi) of Lemma 5, and

$$\lim_{n \to \infty} |\gamma_n| = \infty, \quad \lim_{n \to \infty} |\alpha_n| = 0$$

and

$$\gamma_0 = 0, \ \gamma_n \neq 0 \text{ for } 0 < n < \infty.$$

Also there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that for all $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$ the condition

$$|xu_n + yv_n + zw_n + sr_n| \geq e^{-1} \quad (0 \leq n < \infty)$$

is satisfied.

Proof: Let $(u, v, w, r) \in P_4$ be a fixed element of P_4. Let L be any integer such that

$$e^L \leq e^{-1} |M_1(u, v, w, r)|.$$

Then we claim there are only a finite number of non-zero $(u', v', w', r') \in P_4$ with $\gcd(u', v', w', r') = 1$ satisfying

$$(a) \ |M_1(u, v, w, r)|^2 |M_3(u', v', w', r')|^2 \leq |\Delta|^3,$$

$$(b) \ |M_1(u', v', w', r')| \leq e^L.$$

The proof of this is exactly the same as in Lemma 7.

Now choose (u_0, v_0, w_0, r_0) as in Lemma 5′. We note here that (u_0, v_0, w_0, r_0) as chosen in Lemma 5′ is independent of M and we use same (u_0, v_0, w_0, r_0) for all $\mathfrak{I}_M$.

For each $\mathfrak{I}_M$, the corresponding (u_1, v_1, w_1, r_1) satisfies

$$|M_3(u_1, v_1, w_1, r_1)|^2 |M_1(u_0, v_0, w_0, r_0)|^2 \leq |\Delta|^3$$

and

$$|M_1(u_1, v_1, w_1, r_1)| \leq e^{-1}|\alpha_0|.$$

By the above observation, there are only a finite number of $(u_1, v_1, w_1, r_1) \in P_4$ satisfying above. So we pick out $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1) \in P_4$ which occurs infinitely often. Let $\mathfrak{D}_1$ be the infinite subsequence of those $\mathfrak{I}_M$'s to which $(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)$ belong. Now we consider only these $\mathfrak{I}_M$'s.

Then if M is large and $\mathfrak{I}_M \in \mathfrak{D}_1$, then $\mathfrak{I}_M$ contains more than three terms. Now for each $\mathfrak{I}_M \in \mathfrak{D}_1$,

$$|M_1(u_2, v_2, w_2, r_2)| \leq e^{-1}|M_1(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)|$$

and

$$|M_3(u_2, v_2, w_2, r_2)|^2|M_1(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{r}_1)|^2 \leq |\Delta|^3.$$

As before, there are only a finite number of choices for (u_2, v_2, w_2, r_2). So pick $(\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2) \in P_4$ which occurs in infinitely many $\mathfrak{I}_M$. Let $\mathfrak{D}_2$ be the infinite subsequence of $\mathfrak{D}_1$ of those $\mathfrak{I}_M$'s to which $(\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{r}_2)$ belong.

Continuing this process, we have an infinite sequence $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$ $(0 \leq n < \infty)$ which satisfies conditions (i), (ii), (iii), (vi) of Lemma 5 and further $M_3(\bar{u}_0, \bar{v}_0, \bar{w}_0, \bar{r}_0) = 0$, $M_3(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n) \neq 0$ for $n \neq 0$.
Also by construction

$$\lim_{n \to \infty} |M_1(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)| = 0.$$

As in Lemma 7, we can see that

$$\lim_{n \to \infty} |M_3(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)| = \infty.$$

In the construction of $(\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{r}_n)$'s, we have used subsequences $\mathfrak{D}_1$, $\mathfrak{D}_2, \ldots$ which have the same properties as in Lemma 7.

Following the same method as in Lemma 7, we can get (x_0, y_0, z_0, s_0) with the required properties. In the rest of the proof, we suppress the bar.

Proof of Theorem 3 in Case II.—Let (u_n, v_n, w_n, r_n) $(0 \leq n < \infty)$ and (x_0, y_0, z_0, s_0) be given by Lemma 7′. Suppose a, $b = \tilde{a}$, c, $d = \tilde{c}$ are the values assumed by L_1', L_2', L_3', L_4' respectively for some $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0)$ (mod P_4).

First we assert neither of a, b, c, d is zero. For suppose $a = 0 = b$. Then from (3.1), we get for every $n(0 \leq n < \infty)$,

$$xu_n + yv_n + zw_n + sr_n = \frac{c\gamma_n + d\delta_n}{\Delta'}.$$

So
$xu_0 + yv_0 + zw_0 + sr_0 = 0$ (since $\gamma_0 = 0 = \delta_0$), and this contradicts Lemma 7′.

Now suppose $c = 0 = d$. Then from (3.1) we get that for every n,

$$xu_n + yv_n + zw_n + sr_n = \frac{a\alpha_n + b\beta_n}{\Delta'}$$

or $$|xu_n+yv_n+zw_n+sr_n| = \left|\frac{a\alpha_n+b\beta_n}{\Delta'}\right| \leq \frac{|a\alpha_n|}{|\Delta|}.$$

Making $n \to \infty$, we get

$$\lim_{n\to\infty} |xu_n+yv_n+zw_n+sr_n| = 0.$$

This contradicts Lemma 7′ again.

The proof of Theorem 3 in this case is exactly the same as in earlier case.

§ 6. *Proof of Theorem 3 in Case III.*— $M_1 = 0 = M_2$ for some non-zero $U \in P_4$ and $M_3 = 0 = M_4$ for some other non-zero $U \in P_4$.

Then we must have one of the following:

(*a*) For each $M \geq 1$ there exists an $\mathfrak{S}_M$ with the properties of Lemma 5′; or (*b*) there exists a finite sequence of values α_n, $\beta_n = \tilde{\alpha}_n$, γ_n, $\delta_n = \tilde{\gamma}_n$ of M_1, M_2, M_3, M_4 respectively corresponding to $(u_n, v_n, w_n, r_n) \in P_4$ with gcd $(u_n, v_n, w_n, r_n) = 1$ satisfying conditions (i), (ii), (iii) and (vi) of Lemma 5 and also the following:

(iv′) $\gamma_0 = 0 = \delta_0$, $\gamma_n \neq 0$ for $0 < n \leq N$.

(v′) $\alpha_N = 0 = \beta_N$, $\alpha_n \neq 0$ for $0 \leq n < N$.

The proof of this obviously follows from that of Lemma 5′. If we have (*a*), the rest of the proof is the same as in case II. If we have (*b*), we prove the following lemma.

Lemma 6″.—For the finite sequence obtained above in (*b*), there exists $(x_0, y_0, z_0, s_0) \in R_4$ such that

$$|xu_n+yv_n+zw_n+sr_n| \geq e^{-1} \quad .. \quad .. \quad .. \quad (6.1)$$

for all $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$ and all $n (0 \leq n \leq N)$.

Proof: Since $\alpha_N = 0$, $\gamma_N \neq 0$. So we can choose $\theta = \sigma+\alpha\tau$; $\sigma, \tau \in K\{t\}$ such that

$$\| \gamma_N\theta+\tilde{\gamma}_N\tilde{\theta} \| = e^{-1}.$$

Owing to condition (ii) of Lemma 5, we can apply Lemma 1 with

$$\Lambda_n = \alpha_{N-n}, \mu_n = \gamma_{N-n}\theta+\delta_{N-n}\tilde{\theta} (1 \leq n \leq N) \text{ and we get}$$

$$\xi = \eta+\alpha\zeta \in K^{(m)}\{t\};\ \eta, \zeta \in K\{t\},$$

such that

$$\| \alpha_n\xi+\tilde{\alpha}_n\tilde{\xi}+\gamma_n\theta+\delta_n\tilde{\theta} \| = e^{-1} (0 \leq n < N).$$

Since $\alpha_N = 0$, due to (6.1) we have

$$\| \alpha_N\xi+\beta_N\tilde{\xi}+\gamma_N\theta+\delta_N\tilde{\theta} \| = \| \gamma_N\theta+\tilde{\gamma}_N\tilde{\theta} \| = e^{-1}$$

and

$$\| \alpha_n\xi+\beta_n\tilde{\xi}+\gamma_n\theta+\tilde{\gamma}_n\tilde{\theta} \| = e^{-1} (0 \leq n \leq N). \quad .. \quad .. \quad (6.2)$$

Let x_0, y_0, z_0, s_0 be the solution of

$$L_1(x, y, z, s) = \eta\Delta', \quad L_2(x, y, z, s) = \zeta\Delta',$$
$$L_3(x, y, z, s) = \sigma\Delta', \quad L_4(x, y, z, s) = \tau\Delta'.$$

This (x_0, y_0, z_0, s_0) also satisfies

$$L_1'(x, y, z, s) = \xi\Delta', \quad L_2'(x, y, z, s) = \tilde{\xi}\Delta',$$

$$L_3'(x, y, z, s) = \theta\Delta', \quad L_4'(x, y, z, s) = \tilde{\theta}\Delta'.$$

From (3.1) we get

$$x_0u_n+y_0v_n+z_0w_n+s_0r_n = \xi\alpha_n+\tilde{\xi}\beta_n+\theta\gamma_n+\tilde{\theta}\delta_n.$$

Then from (6.2) we get

$$||x_0u_n+y_0v_n+z_0w_n+s_0r_n|| = ||\xi\alpha_n+\tilde{\xi}\beta_n+\theta\gamma_n+\tilde{\theta}\delta_n|| = e^{-1}$$
$$(0 \leq n \leq N).$$

Take $(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}$.
Then

$$|xu_n+yv_n+zw_n+sr_n| \geq ||xu_n+yv_n+zw_n+sr_n||$$
$$= ||x_0u_n+y_0v_n+z_0w_n+s_0r_n|| = e^{-1} \ (0 \leq n \leq N).$$

Proof of Theorem 3.—Let $(x_0, y_0, z_0, s_0) \in R_4$ be given by Lemma 6″. Take

$$(x, y, z, s) \equiv (x_0, y_0, z_0, s_0) \pmod{P_4}.$$

Let $a, b, = \tilde{a}, c, d = \tilde{c}$ be the values taken by

$$L_1', L_2', L_3', L_4'$$

respectively at (x, y, z, s). As in case II, we can suppose $a \neq 0 \neq b$. Now suppose $c = 0 = d$. From (3.1) we get for every n

$$xu_n+yv_n+zw_n+sr_n = \frac{a\alpha_n+b\beta_n}{\Delta'}.$$

So

$$xu_N+yv_N+zw_N+sr_N = 0.$$

This contradicts Lemma 6″.

Now as in (4.5), if we can choose an integer $n(1 \leq n \leq N)$ such that

$$|\gamma_{n-1}|^2 \leq |\Delta|^{\frac{1}{2}}\left|\frac{a}{c}\right| \leq |\gamma_n|^2,$$

the proof is similar as in case I. If not, we have

$$|\gamma_N|^2 < |\Delta|^{\frac{1}{2}}\left|\frac{a}{c}\right|.$$

But from (3.1) and using $\alpha_N = 0 = \beta_N$, we get

$$|xu_N+yv_N+zw_N+sr_N| = \frac{|c\gamma_N+d\delta_N|}{|\Delta'|} \leq \frac{|c\gamma_N|}{|\Delta|}.$$

So

$$|xu_N+yv_N+zw_N+sr_N|^2 \leq \frac{|c^2||\gamma_N|^2}{|\Delta|^2} \leq \frac{|ac|}{|\Delta|^{\frac{1}{2}}}.$$

But by Lemma 6″, we get

$$|xu_N+yv_N+zw_N+sr_N| \geq e^{-1}.$$

Hence

$$\frac{|ac|}{|\Delta|^{\frac{1}{2}}} \geq e^{-2} \text{ or } |a^2c^2| \geq \frac{|\Delta|}{e^4} \text{ or } |abcd| \geq \frac{|\Delta|}{e^4}.$$

This proves the theorem in this case.

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References

Armitage, J. V. (1957*a*). Euclid's algorithm in certain algebraic function fields. *Proc. Lond. math. Soc.*, 7, 498–509.

——— (1957*b*). The product of n linear forms in a field of series. *Mathematika*, 4, 132–137.

Cassels, J. W. S. (1952*a*). The inhomogeneous minimum of binary quadratic, ternary cubic and quaternary quartic forms. *Proc. Camb. Phil. Soc.*, 48, 72–86.

——— (1952*b*). Addendum to the paper mentioned above. *Proc. Camb. Phil. Soc.*, 48, 519-520.

Davenport, H. (1950*a*). Euclid's algorithm in cubic fields of negative discriminant. *Acta math.*, 84, 159–179.

——— (1950*b*). Euclid's algorithm in certain quartic fields. *Trans. Am. math. Soc.*, 68, 508–532.

——— (1951). Indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields. *Proc. Lond. math. Soc.*, 53, 65–82.

Mahler, K. (1941). An analogue to Minkowski's geometry of numbers in a field of series. *Ann. math.*, 42, 488–522.

RADIAL OSCILLATIONS OF COMPOSITE POLYTROPES—PART II

by MANMOHAN SINGH, *Department of Mathematics, University of Roorkee*

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Small radial oscillations of four composite polytropic models with interfaces $\xi_i = 0{\cdot}2094$, $0{\cdot}3915$, $0{\cdot}6160$ and $0{\cdot}8250$, having indices $1{\cdot}5$ in the core and 3 in the envelope, have been considered. The anharmonic pulsations of these models have also been considered.

1. INTRODUCTION

We consider in this paper the small adiabatic radial oscillations of the composite polytropes having indices $1{\cdot}5$ in the core and 3 in the envelope. Four such models with interfaces at $\xi_i = 0{\cdot}2094$, $0{\cdot}3915$, $0{\cdot}6160$ and $0{\cdot}8250$ have been obtained in the Part I [Manmohan Singh (1969)]. The radial oscillations of these models have been worked out for the first four modes taking $\alpha = 0{\cdot}6$, $0{\cdot}5$ and $0{\cdot}4$. We have also considered their anharmonic pulsations for the case $\alpha = 0{\cdot}6$.

2. SMALL RADIAL OSCILLATIONS

Using Emden variables ξ and ϕ, the equations of pulsation for the core ($n = 1{\cdot}5$) may be written as

$$\frac{d^2\eta}{d\xi^2}+\frac{1}{\xi}\left(4+\frac{5}{2}\frac{\xi}{\phi}\frac{d\phi}{d\xi}\right)\frac{d\eta}{d\xi}+\left(\frac{\omega^2}{\phi}+\frac{5}{2}\frac{\alpha}{\xi\phi}\frac{d\phi}{d\xi}\right)\eta=0 \qquad .. \quad .. \quad (1)$$

where

$$\frac{\omega^2}{A^4}=\frac{R^2\sigma^2\rho_c}{\gamma P_c} \qquad .. \quad .. \quad .. \quad .. \quad (2)$$

and other symbols have the same meanings as given by Prasad and Gurm (1961).

For the envelope ($n = 3$) the equations of pulsation in terms of variables ξ and θ are

$$\frac{d^2\eta}{d\xi^2}+\frac{4}{\xi}\left(1+\frac{\xi}{\theta}\frac{d\theta}{d\xi}\right)\frac{d\eta}{d\xi}+\left(\frac{\Omega^2}{\theta}+\frac{4\alpha}{\xi\theta}\frac{d\theta}{d\xi}\right)\eta=0 \qquad .. \quad .. \quad (3)$$

Here

$$\Omega^2 = \frac{R^2\sigma^2\rho_c}{\gamma P_c}\frac{\theta_i}{\phi_i}$$

$$= \frac{\omega^2}{A^4}\frac{\theta_i}{\phi_i} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (4)$$

where θ_1 and ϕ_1 are the values of θ and ϕ at the interface.

Eqn. (1) has singularities at the centre ($\xi = 0$) and also at the surface. But as the solution of this equation does not extend to the surface, we only require a solution finite at $\xi = 0$. So we develop a solution in series near the centre which is non-singular at the centre $\xi = 0$.

With the help of well-known expansion [B.A.M. Tables (1932)] for ϕ we have

$$\phi = A^4\left\{1-\frac{A^2\xi^2}{6}+\frac{A^4\xi^4}{80}-\frac{A^6\xi^6}{1440}+\ldots\right\} \quad .. \quad .. \quad (5)$$

The solution in series for the centre near $\xi = 0$ is assumed to be

$$\eta = c_0+c_2\xi^2+c_4\xi^4+c_6\xi^6+\ldots \quad .. \quad .. \quad .. \quad (6)$$

On substitution in eqn. (1) we can find the coefficients c_2, c_4, c_6. . . . by equating the coefficients of various powers of ξ.

The eqn. (3) for the envelope has regular singularities at $\xi = 0$ and 1. A solution in series started from one extremity usually becomes divergent at the other. For the envelope we require a solution finite at $\xi = 1$ and not extending to the centre. To obtain it we put $\xi = 1-t$ in eqn. (3). We have

$$\frac{d^2\eta}{dt^2}-\frac{4}{1-t}\left(1-\frac{1-t}{\theta}\frac{d\theta}{dt}\right)\frac{d\eta}{dt}+\left(\frac{\Omega^2}{\theta}-\frac{4\alpha}{1-t}\frac{1}{\theta}\frac{d\theta}{dt}\right)\eta = 0 \quad .. \quad .. \quad (7)$$

To envelope a series solution we have taken θ as follows:

$$\theta = \omega_n\left\{t+t^2+t^3+t^4+\left(1-\frac{1}{20}\omega_n^2\right)t^5+\ldots\right\} \quad .. \quad .. \quad (8)$$

where

$$\omega_n = -\left(\frac{d\theta}{d\xi}\right)_{\xi=1} \quad .. \quad .. \quad .. \quad .. \quad (9)$$

and the series assumed is

$$\eta = 1+a_1t+a_2t^2+a_3t^3+\ldots \quad .. \quad .. \quad .. \quad (10)$$

On substituting (8) and (10) in (7) we can find the coefficients a_1, a_2, a_3, . . . by equating to zero the coefficients of various power of t.

The solutions of eqns. (1) and (7), satisfying boundary conditions at the two extremities as well as the interface, have been found out in the same way as by Prasad (1953).

The values of ω^2/A^4 for the first four modes for three values of α (= 0·6, 0·5 and 0·4) are given for the four models in Table I.

TABLE I

Values of ω^2/A^4

Model with interface at	α	zero-th mode	1st mode	2nd mode	3rd mode
$\xi = 0{\cdot}2094$	0·6	6·023504	11·226409	19·105066	29·279197
	0·5	5·306420	10·560777	18·494241	28·689014
	0·4	4·481545	9·916510	17·899210	28·108813
$\xi = 0{\cdot}3915$	0·6	5·554554	11·919778	21·318248	33·279639
	0·5	4·744743	11·402158	20·829913	32·793357
	0·4	3·884831	10·903206	20·348435	32·309936
$\xi = 0{\cdot}6160$	0·6	4·396333	14·232968	25·697994	39·415017
	0·5	3·693355	13·807355	25·247873	38·919717
	0·4	2·978025	13·382857	24·797359	38·428582
$\xi = 0{\cdot}8250$	0·6	3·498776	15·719071	29·998478	46·498867
	0·5	2·931580	15·239002	29·561448	46·048814
	0·4	2·358007	14·759121	29·124757	45·599013

3. ANHARMONIC PULSATIONS

For small radial oscillations the square of the amplitude is neglected in the equation of motion. This gives a symmetrical sine curve for the radial velocity of pulsating stars. But observation shows that the 'Cepheid Variables' have velocity curves, which are far from being represented by a simple sine curve. They usually show a steep rise to a maximum and then a slower decline to minimum. This necessitates the consideration of the second order terms.

The equations of anharmonic pulsations (Prasad 1949*b*) for the composite polytrope may be written as

$$\frac{d^2q}{d^2}+\beta_m q_m = \frac{A^4}{\omega_1^2 I_m}\left\{\sum_{jj} D_{jjm}q_j^2+2\sum_{jk} D_{jkm}q_j q_k\right\} \quad .. \quad .. \quad (11)$$

where

$$r = R\xi \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (12)$$

$$\tau = \sigma_1 t \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (13)$$

$$\beta_m = \frac{\sigma_m^2}{\sigma_1^2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (14)$$

$$I_m = \int_0^{\xi_i} \phi^{3/2}\xi^4\eta_m^2\, d\xi + \frac{\phi_i^{3/2}}{\theta_i^3}\int_{\xi_i}^1 \theta^3\xi^4\eta_m^2\, d\xi \quad .. \quad .. \quad .. \quad .. \quad (15)$$

and

$$D_{jkm} = -\frac{1}{2}\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\left\{\frac{5}{2}\int_0^{\xi_i}\phi^{3/2}\phi'\xi^3\eta_j\eta_k\eta_m\,d\xi\right.$$

$$\left.+4\frac{\phi_i^{5/2}}{\phi_i^4}\int_{\xi_i}^1\theta^3\theta'\xi^3\eta_j\eta_k\eta_m\,d\xi\right\}$$

$$+\frac{3\gamma-1}{2}\left\{\int_0^{\xi_i}\phi^{5/2}\xi^4\left(\eta_j\eta_k'\eta_m'+\eta_j'\eta_k\eta_m'+\eta_j'\eta_k'\eta_m\right)d\xi\right.$$

$$\left.+\frac{\phi_i^{5/2}}{\theta_i^4}\int_{\xi_i}^1\theta^4\xi^4\left(\eta_j\eta_k'\eta_m'+\eta_j'\eta_k\eta_m'+\eta_j'\eta_k'\eta_m\right)d\xi\right\}$$

$$+\frac{\gamma-1}{2}\left\{\int_0^{\xi_i}\phi^{5/2}\xi^5\eta_j\eta_k\eta_m\,d\xi+\frac{\phi_i^{5/2}}{\theta_i^4}\int_{\xi_i}^1\theta^4\xi^5\eta_j'\eta_k'\eta_m'\,d\xi\right\} \quad .. \quad (16)$$

where j, k, m are the positive integral values starting with unity, q's are unknown functions of time, η's are the relative amplitudes normalized to unity at the surface of the star and $2\pi/\sigma_m$ are the periods for small oscillations, ξ_i, θ_i and ϕ_i are the values of ξ, θ and ϕ at the interface.

I's and D's are evaluated by numerical integration. This gives the following differential equations for the first three modes:

(i) $\xi_i = 0{\cdot}2094$

$$\left.\begin{aligned}
&\frac{d^2q_1}{d\tau^2}+q_1 = 0{\cdot}952650\ q_1^2+0{\cdot}593014\ q_1q_2+0{\cdot}069846\ q_1q_3\\
&\frac{d^2q_2}{d\tau^2}+1{\cdot}863767\ q_2 = 4{\cdot}982256\ q_1^2+7{\cdot}873256\ q_1q_2+2{\cdot}463878\ q_1q_3\\
&\frac{d^2q_3}{d\tau^2}+3{\cdot}171753\ q_3 = 5{\cdot}144444\ q_1^2+21{\cdot}600000\ q_1q_2+19{\cdot}511111\ q_1q_3
\end{aligned}\right\} \quad .. \quad (17)$$

(ii) $\xi_i = 0{\cdot}3915$

$$\left.\begin{aligned}
&\frac{d^2q_1}{d\tau^2}+q_1 = 0{\cdot}874390\ q_1^2+0{\cdot}370095\ q_1q_2+0{\cdot}005027\ q_1q_3\\
&\frac{d^2q_2}{d\tau^2}+2{\cdot}145947\ q_2 = 6{\cdot}514288\ q_1^2+9{\cdot}596708\ q_1q_2+2{\cdot}338361\ q_1q_3\\
&\frac{d^2q_3}{d\tau^2}+3{\cdot}837979\ q_3 = 0{\cdot}809235\ q_1^2+21{\cdot}908740\ q_1q_2+21{\cdot}827020\ q_1q_3
\end{aligned}\right\} \quad .. \quad (18)$$

(iii) $\xi_i = 0{\cdot}6160$

$$\left.\begin{aligned}
&\frac{d^2q_1}{d\tau^2}+q_1 = 0{\cdot}906197\ q_1^2+0{\cdot}264662\ q_1q_2-0{\cdot}029630\ q_1q_3\\
&\frac{d^2q_2}{d\tau^2}+3{\cdot}237463\ q_2 = 11{\cdot}073776\ q_1^2+17{\cdot}342202\ q_1q_2+3{\cdot}174770\ q_1q_3\\
&\frac{d^2q_3}{d\tau^2}+5{\cdot}845325\ q_3 = -11{\cdot}054628\ q_1^2+28{\cdot}307066\ q_1q_2+33{\cdot}9788648\ q_1q_3
\end{aligned}\right\} \quad (19)$$

(iv) $\xi_i = 0{\cdot}8250$

$$\left.\begin{aligned}
&\frac{d^2q_1}{d\tau^2}+q_1 = 0{\cdot}706618\ q_1^2+0{\cdot}319456\ q_1q_2-0{\cdot}014567\ q_1q_3\\
&\frac{d^2q_2}{d\tau^2}+4{\cdot}492734\ q_2 = 7{\cdot}379650\ q_1^2+18{\cdot}095526\ q_1q_2+0{\cdot}180129\ q_1q_3\\
&\frac{d^2q_3}{d\tau^2}+8{\cdot}573992\ q_3 = -4{\cdot}044655\ q_1^2+21{\cdot}650820\ q_1q_2+38{\cdot}715066\ q_1q_3
\end{aligned}\right\} \quad (20)$$

The method of the solution of these equations is that given by Prasad (1949*a*). The solutions of eqns. (17), (18), (19) and (20) have been started by taking $a_1 = 0{\cdot}06$. The radial velocity curves for the four cases are shown in Fig. 1.

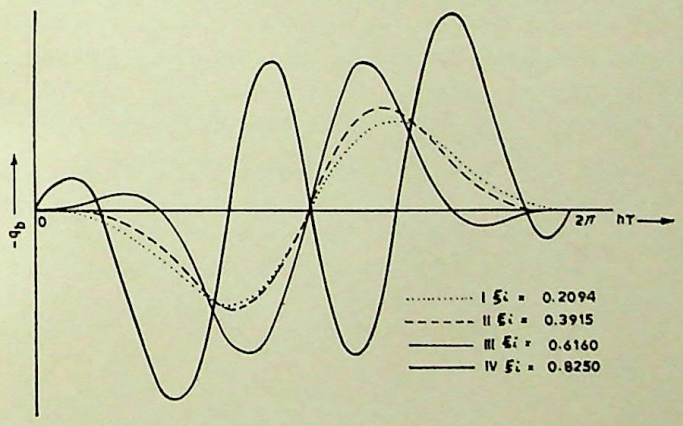

FIG. 1. The radial velocity curves for the composite polytropes.

4. CONCLUSIONS

From the results of § 2 we conclude that the composite model with interface $\xi_i = 0{\cdot}2094$ is quite near to the standard model in the pulsational behaviour, the composite model with interface $\xi_i = 0{\cdot}3915$ has drifted away from the standard model and the composite model with interface $\xi_i = 0{\cdot}6160$ is very near to the complete polytrope $n = 2$. The last composite model with interface $\xi_i = 0{\cdot}8250$ has shifted closer to the complete polytrope $n = 1{\cdot}5$.

As for the anharmonic pulsations, we see from Fig. 1 that radial velocity curves of model with interface at $\xi_i = 0{\cdot}2094$ and $0{\cdot}3915$ lie very near to each other and they have skewness near that of the standard model. In the last two models with interface at $\xi_i = 0{\cdot}6160$ and $0{\cdot}8250$ the radial velocity curves show humps and so we regard these two models unsuitable as far as their anharmonic pulsations are concerned.

The comparison of the pulsation characteristics of these models with some other models has been given in another paper by the author (1968).

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References

British Association Mathematical Tables, Vol. 2 (1932). Cambridge University Press, London.

Prasad, C. (1949*a*). On the anharmonic pulsations of the standard model. *Mon. Not. R. astr. Soc.*, **109**, 528.

——— (1949*b*). Anharmonic pulsations of two particular models. *Astrophys. J.*, **110**, 375.

——— (1953). Radial oscillations of a composite model. *Proc. natn. Inst. Sci., India*, **19**, 739.

Prasad, C., and Gurm, H. S. (1961). Radial pulsations of polytrope $n = 2$. *Mon. Not. R. astr. Soc.*, **122**, 409.

Singh, Manmohan (1968). Effect of central condensation on the pulsation characteristics. *Mon. Not. R. astr. Soc.*, **140**, 235–240.

——— (1969). Radial oscillations of composite polytropes—Part I. *Proc. natn. Inst. Sci., India*, **35A**, 586–589.

ON MECHANICAL RESPONSE IN A PIEZOELECTRIC ANNULAR DISK TRANSDUCER OWING TO A STEP INPUT WITH A PRESCRIBED TEMPERATURE FIELD

by B. Chaudhuri, *Department of Mathematics, Jadavpur University, Calcutta*

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The methods of transform calculus are used as tools to calculate the mechanical response in a piezoelectric annular disk owing to a step voltage with a prescribed temperature field.

Introduction

It is sufficiently well known that the piezoelectric transducers play an important role in the generation and detection of ultrasonic waves, vide Mason (1950), Cady (1946). The problems of producing mechanical response by a voltage step in a piezoelectric transducer, in the form of a plate and bar, have been investigated by Redwood (1961*a*, *b*), Sinha (1962) and Giri (1966) and their discussions are confined to the plate and the bar transducers. Sinha (1965) in his recent paper has studied the mechanical response of a piezoelectric annular disk transducer owing to an impulsive voltage input. The present paper is an attempt to study the mechanical response in a piezoelectric transducer in the form of an annular disk owing to a step voltage input with a prescribed temperature field. The solution has been facilitated by the use of Laplace transforms and under suitable assumptions similar to those of Redwood (1961*a*).

Problem, Fundamental Equations and Boundary Conditions

For the present case we need an annular disk transducer of piezoelectric material. Let r_1 and r_2 be the inner and outer radii of the boundary of the disk. Let V be the electrical voltage in the form of a step function. Let the temperature at the inner boundary $r = r_1$ be T_1 and the outer boundary $r = r_2$ be at zero temperature. Now our problem is to investigate the nature of the mechanical response in the above type of transducer.

To serve the purpose of our problem we first need a relation connecting the variables representing mechanical, electrical and thermal fields. We choose the mechanical (radial) displacement u as the mechanical variable, the electrical voltage V as the electrical variable and the temperature T as the thermal variable. To simplify the algebra, we assume the disk to be open-circuited. Referred to the usual coordinates, the equations of piezoelectricity as suggested by Mason (1950) have been modified for the present problem and they are given by

$$\sigma_r = C_{11}\frac{du}{dr}+C_{12}\frac{u}{r}-hD_r-\lambda_1 T \quad .. \quad .. \quad .. \quad (1)$$

$$\sigma_\theta = C_{12}\frac{du}{dr}+C_{11}\frac{u}{r}-hD_r-\lambda_2 T \quad .. \quad .. \quad .. \quad (2)$$

$$E_r = -h\left(\frac{du}{dr}-\frac{u}{r}\right)+\frac{D_r}{\epsilon}-\nu T \quad .. \quad .. \quad .. \quad (3)$$

where $(\sigma_r, \sigma_\theta)$ are the components of stress, D_r is the radial component of the electric displacement $\vec{D}$, E_r the radial component of the electric intensity $\vec{E}$, C_{11}, C_{12} the elastic compliances, h the piezoelectric constant, ϵ the permittivity and T the temperature.

The equation of motion in the radial direction is given by

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}=\rho\frac{\partial^2 u}{\partial t^2}. \quad .. \quad .. \quad .. \quad .. \quad (4)$$

Gauss's equation given by

$$\operatorname{div}\vec{D}=0$$

in polar coordinates (r, θ) becomes

$$\frac{\partial D_r}{\partial r}+\frac{1}{r}\frac{\partial D_\theta}{\partial \theta}=0.$$

Assuming the disturbances to propagate in the radial direction, we have from the last equation,

$$\frac{\partial D_r}{\partial r}=0. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

The equation of steady heat flow to be satisfied by the temperature T is

$$\nabla^2 T=0. \quad .. \quad .. \quad .. \quad .. \quad (6)$$

The solution of eqn. (6), subject to the boundary conditions

$$\left.\begin{aligned} T &= T_1 \text{ on } r = r_1 \\ T &= 0 \quad \text{on } r = r_2 \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad (7)$$

is

$$T = \frac{T_1}{\log \frac{r_2}{r_1}} \log \frac{r_2}{r}.$$

Considering eqns. (1), (2), (4), (5) and (7), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\lambda_1}{C_{11}} \cdot \frac{T_1}{\log \frac{r_2}{r_1}} \cdot \frac{1}{r} + \frac{(\lambda_2 - \lambda_1)}{C_{11} r} \cdot \frac{T_1}{\log \frac{r_2}{r_1}} \log \frac{r_2}{r} = \frac{\rho}{C_{11}} \frac{\partial^2 u}{\partial t^2}. \quad (8)$$

Taking Laplace transform defined by

$$\bar{u}(r, p) = \int_0^\infty u(r, t) e^{-pt} dt, \quad p > 0$$

we have

$$\frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \left(\frac{1}{r^2} + p^2\alpha^2\right)\bar{u} = \left(K_2 + K_2' \log \frac{r_2}{r}\right) \cdot \frac{1}{rp} \quad .. \quad .. \quad (9)$$

where

$$\alpha = \left(\frac{\rho}{C_{11}}\right)^{\frac{1}{2}}$$

$$K_2 = -\frac{\lambda_1}{C_{11}} \cdot \frac{T_1}{\log \frac{r_2}{r_1}}$$

$$K_2' = \frac{\lambda_1 - \lambda_2}{C_{11}} \cdot \frac{T_1}{\log \frac{r_2}{r_1}}.$$

Solving eqn. (9), we get

$$\bar{u} = AI_1(\alpha pr) + BK_1(\alpha pr) + \frac{K_2' C_{11}}{\rho r p^3} \log r - \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho p^3 r} \quad .. \quad (10)$$

where I_1 and K_1 are Bessel functions of imaginary argument and of order one, A and B are constants to be determined from boundary conditions. From eqn. (3)

$$\bar{E}_r = -h\left(\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r}\right) + \frac{\bar{D}_r}{\epsilon} - \frac{\nu T_1}{p \log \frac{r_2}{r_1}} \log \frac{r_2}{r}.$$

Therefore the potential $\bar{V}$ across the disk is given by

$$\bar{V} = \int_{r_1}^{r_2} \bar{E}_r dr = -h[(\bar{u})_{r=r_2} - (\bar{u})_{r=r_1}] - \frac{\nu T_1}{p \log \frac{r_2}{r_1}} (r_2 - r_1) \log r_2$$

$$+ \frac{\nu T_1}{p \log \frac{r_2}{r_1}} (r_2 \log r_2 - r_2 - r_1 \log r_1 + r_1) + h \int_{r_1}^{r_2} \frac{\bar{u}}{r} dr \quad .. \quad .. \quad (11)$$

where we have considered the fact that the disk is open-circuited (so that $\bar{Q}$, the surface charge is zero and hence $\bar{D}_r$ related to $\bar{Q}$ is zero). Now eqns. (10) and (11) constitute the fundamental equations of the problem.

With the value of $\bar{u}$ from eqns. (10) and (11) can also be put in the following form:

$$\bar{V} = -h\left[AI_1(\alpha p r_2) + BK_1(\alpha p r_2) + \frac{K_2' C_{11}}{\rho r_2 p^3} \log r_2 - \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho r_2 p^3}\right.$$

$$\left. - AI_1(\alpha p r_1) - BK_1(\alpha p r_1) - \frac{K_2' C_{11}}{\rho r_1 p^3} \log r_1 + \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho r_1 p^3}\right]$$

$$- \frac{\nu T_1 (r_2 - r_1) \log r_2}{p \log \frac{r_2}{r_1}} + \frac{\nu T_1}{p \log \frac{r_2}{r_1}} (r_2 \log r_2 - r_2 - r_1 \log r_1 + r_1)$$

$$+ h\left[\int_{r_1}^{r_2} \left\{A \frac{I_1(\alpha p r)}{r} + B \frac{K_1(\alpha p r)}{r}\right\} dr\right]$$

$$+ \frac{h K_2' C_{11}}{\rho p^3} \left(\frac{\log r_1}{r_1} - \frac{\log r_2}{r_2}\right) + \frac{(K_2' - K_2 - K_2' \log r_2) h C_{11}}{\rho p^3} \left(\frac{1}{r_1} - \frac{1}{r_2}\right). \quad .. \quad (12)$$

In accordance with our assumption, we have

$$V = V_0 H(t) \quad .. \quad .. \quad .. \quad .. \quad (13)$$

where $H(t)$ is Heaviside unit function equal to unity when $t > 0$ and equal to zero when $t < 0$. We, therefore, have

$$\bar{V} = \frac{V_0}{p}. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (14)$$

In addition to the condition (13), we assume the boundary of the disk to be rigidly fixed, so that

$$(\bar{u})_{r=r_2} = 0. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (15)$$

Because of eqns. (14) and (15), it follows from eqns. (10) and (12) that

$$AI_1(\alpha p r_2) + BK_1(\alpha p r_2) = -\frac{L}{p^3} \quad .. \quad .. \quad .. \quad (16)$$

$$Aa_{11} + Ba_{12} = -\frac{M}{p^3} + \frac{(V_0 + N)}{p} \quad .. \quad .. \quad .. \quad (17)$$

where

$$a_{11} = h\left[I_1(\alpha p r_1) + \int_{r_1}^{r_2} \frac{I_1(\alpha p r)}{r}\, dr\right] \quad .. \quad .. \quad .. \quad (18)$$

$$a_{12} = h\left[K_1(\alpha p r_1) + \int_{r_1}^{r_2} \frac{K_1(\alpha p r)}{r}\, dr\right] \quad .. \quad .. \quad (19)$$

$$L = \frac{K_2' C_{11} \log r_2}{\rho r_2} - \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho r_2}$$

$$M = \frac{2hK_2' C_{11} \log r_1}{\rho r_1} - \frac{h(K_2 + K_2' \log r_2) C_{11}}{\rho r_1} - \frac{hK_2' C_{11} \log r_2}{\rho r_2}$$

$$+ \frac{h(K_2' - K_2 - K_2' \log r_2) C_{11}}{\rho} \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$N = \frac{\nu T_1 (r_2 - r_1) \log r_2}{\log \frac{r_2}{r_1}} - \frac{\nu T_1}{\log \frac{r_2}{r_1}} (r_2 \log r_2 - r_2 - r_1 \log r_1 + r_1).$$

Solving eqns. (16) and (17) we get

$$A = \frac{[MK_1(\alpha p r_2) - La_{12}] \frac{1}{p^3} - \frac{1}{p}(V_0 + N) K_1(\alpha p r_2)}{[a_{12} I_1(\alpha p r_2) - a_{11} K_1(\alpha p r_2)]} \quad .. \quad .. \quad .. \quad (20)$$

$$B = \frac{[MI_1(\alpha p r_2) - La_{11}] \frac{1}{p^3} - \frac{1}{p}(V_0 + N) I_1(\alpha p r_2)}{[a_{11} K_1(\alpha p r_2) - a_{12} I_1(\alpha p r_2)]} . \quad .. \quad .. \quad .. \quad (21)$$

Substituting these values of A and B in eqn. (10) and then taking the inverse transform of the resulting expression, we get the mechanical displacement u corresponding to V given by condition (13). From this, the displacement of the inner boundary $r = r_1$ can also be calculated. This displacement, from eqn. (10), is given by

$$(\bar{u})_{r = r_1} = AI_1(\alpha p r_1) + BK_1(\alpha p r_1) + \frac{W}{p^3}$$

where

$$W = \frac{K_2' C_{11} \log r_1}{\rho r_1} - \frac{(K_2 + K_2' \log r_2) C_{11}}{\rho r_1} .$$

By eqns. (20) and (21)

$$(\bar{u})_{r = r_1} = \frac{1}{[a_{12} I_1(\alpha p r_2) - a_{11} K_1(\alpha p r_2)] p^3} [\{MK_1(\alpha p r_2) - La_{12} - (V_0 + N) p^2 K_1(\alpha p r_2)\}$$

$$I_1(\alpha p r_1) - \{MI_1(\alpha p r_2) - La_{11} - (V_0 + N) p^2 I_1(\alpha p r_2)\} K_1(\alpha p r_1)] + \frac{W}{p^3} .$$

Inverting the Laplace transform, as given by Carslaw and Jaeger (1959), we have

$$(u)_{r=r_1} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{pt}[\{MK_1(\alpha pr_2) - La_{12} - (V_0+N)p^2K_1(\alpha pr_2)\}I_1(\alpha pr_1) - \{MI_1(\alpha pr_2) - La_{11} - (V_0+N)p^2I_1(\alpha pr_2)\} \times K_1(\alpha pr_1)]}{p^3[a_{12}I_1(\alpha pr_2) - a_{11}K_1(\alpha pr_2)]} + \frac{Wt^2}{2} \cdot \quad .. \quad (22)$$

Now putting the value of a_{11} and a_{12} from eqns. (18) and (19) it follows from eqn. (22) that

$$(u)_{r=r_1} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{-Me^{pt}[K_1(\alpha pr_2)I_1(\alpha pr_1) - I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(V_0+N)e^{pt}[K_1(\alpha pr_2)I_1(\alpha pr_1) - I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{Le^{pt}\left[I_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr - K_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr\right]}{p^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right] + \frac{Wt^2}{2}}$$

or

$$(u)_{r=r_1} = \frac{1}{2\pi i}(S_1+S_2+S_3) + \frac{Wt^2}{2} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (23)$$

S_1 is $2\pi i$ times the sum of the residues of the function

$$\frac{-Me^{pt}[K_1(\alpha pr_2)I_1(\alpha pr_1) - I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

lying to the left of the line $R(p) = \gamma$. The poles of this are at $p = 0$ and at the zeros of

$$f(p) = \left[I_1(\alpha pr_1)K_1(\alpha pr_2) - I_1(\alpha pr_2)K_1(\alpha pr_1) + K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr - I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right].$$

These zeros are at $p = i\alpha_n/\alpha$, as in Carslaw and Jaeger (1959), where $\pm\alpha_n$ ($n = 1, 2, 3, \ldots$) are the roots of

$$\left[J_1(\alpha_n r_1)\{iy_1(\alpha_n r_2)-J_1(\alpha_n r_2)\}-J_1(\alpha_n r_2\{iy_1(\alpha_n r_1)-J_1(\alpha_n r_1)\}+\{iy_1(\alpha_n r_2)-J_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr-J_1(\alpha_n r_2)\int_{r_1}^{r_2}\frac{\{iy_1(\alpha_n r_1)-J_1(\alpha_n r_1)\}}{r}\,dr\right] = 0$$

where J_1 and Y_1 are Bessel functions of first order. The residue at $p = 0$ is zero. The residue at $p = i\alpha_n/\alpha$ is

$$\left[p\frac{d}{dp}\left\{hI_1(\alpha p r_1)K_1(\alpha p r_2)-hI_1(\alpha p r_2)K_1(\alpha p r_1)+hK_1(\alpha p r_2)\int_{r_1}^{r_2}\frac{I_1(\alpha p r)}{r}\,dr-hI_1(\alpha p r_2)\int_{r_1}^{r_2}\frac{K_1(\alpha p r)}{r}\,dr\right\}\right]_{p=id_n/\alpha} = hi\frac{\pi}{2}(A_1+iB_1)$$

where

$$A_1 = \left[\{J_1(\alpha_n r_2)-\alpha_n r_2 J_0(\alpha_n r_1)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr+\{\alpha_n r_2 J_0(\alpha_n r_2)-J_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{y_1(\alpha_n r)}{r}\,dr\right]$$

$$B_1 = \Big[\alpha_n r_1\{J_0(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_0(\alpha_n r_1)\}-\alpha_n r_2\{J_0(\alpha_n r_2)Y_1(\alpha_n r_1)-J_1(\alpha_n r_1)Y_0(\alpha_n r_2)\}+3\{J_1(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_1(\alpha_n r_1)\}$$
$$+\{\alpha_n r_2 Y_0(\alpha_n r_2)-Y_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr+\{\alpha_n r_2 J_0(\alpha_n r_2)-J_1(\alpha_n r_2)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}\,dr\Big].$$

Similar results for other two integrals S_2 and S_3 can be obtained. Hence from eqn. (23)

$$(u)_{r=r_1} = \text{Real part of}\left\{\sum_{n=1}^{\infty}\frac{-Mie^{\frac{id_n t}{\alpha}}(A_1-iB_1)[J_1(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_1(\alpha_n r_1)]}{h(A_1^2+B_1^2)}\right\}$$
$$+\text{Real part of}\left\{\sum_{n=1}^{\infty}\frac{(V_0+N)ie^{\frac{i\alpha_n t}{\alpha}}(A_1-iB_1)[J_1(\alpha_n r_1)Y_1(\alpha_n r_2)-J_1(\alpha_n r_2)Y_1(\alpha_n r_1)]}{h(A_1^2+B_1^2)}\right\}$$

$$+\text{ Real part of }\left\{\sum_{n=1}^{\infty}\frac{Le^{\frac{i\alpha_n t}{\alpha}}(A_1-iB_1)\left[J_1(\alpha_n r_1)\int_{r_1}^{r_2}\frac{\{-J_1(\alpha_n r)+iY_1(\alpha_n r)\}}{r}dr+\{-J_1(\alpha_n r_1)+iY_1(\alpha_n r_1)\}\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}dr\right]}{(A_1^2+B_1^2)}\right\}+\frac{Wt^2}{2}. \quad (24)$$

Similarly in this manner, the radial pressure or circumferential tension can be evaluated. But these evaluations as well as the previous one involve the evaluation of the definite integrals

$$\int_{r_1}^{r_2}\frac{J_1(\alpha_n r)}{r}dr \quad \text{and} \quad \int_{r_1}^{r_2}\frac{Y_1(\alpha_n r)}{r}dr\,.$$

By using the series expansions for $J_1(\alpha_n\, r)$ and $Y_1(\alpha_n\, r)$ (*see* Erdeyli 1953), these integrals can be evaluated.

Approximate Evaluation of the Displacement of the Inner Boundary $r = r_1$ for Large Values of Time

$$(\bar{u})_{r=r_1}=\frac{-M[K_1(\alpha pr_2)I_1(\alpha pr_1)-I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

$$+\frac{(V_0+N)[K_1(\alpha pr_2)I_1(\alpha pr_1)-I_1(\alpha pr_2)K_1(\alpha pr_1)]}{hp\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}$$

$$+\frac{L\left[I_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr-K_1(\alpha pr_1)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr\right]}{p^3\left[I_1(\alpha pr_1)K_1(\alpha pr_2)-I_1(\alpha pr_2)K_1(\alpha pr_1)+K_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{I_1(\alpha pr)}{r}dr-I_1(\alpha pr_2)\int_{r_1}^{r_2}\frac{K_1(\alpha pr)}{r}dr\right]}+\frac{W}{p^3}. \quad \ldots \quad \ldots \quad (25)$$

Its evaluation for large values of time will be possible if we expand it in powers of p [*see* Carslaw and Jaeger (1959)]. p is small for large values of time. Therefore, in the expansion of the relevant quantities in powers of p, we neglect squares and higher powers of p.

The expansions for $I_1(\alpha pr)$ and $K_1(\alpha pr)$ are given by Carslaw and Jaeger (1959).

$$I_1(\alpha pr) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}\alpha pr)^{2m+1}}{m!\ \Gamma(m+2)}$$

$$K_1(\alpha pr) = \{\log(\tfrac{1}{2}\alpha pr)+\gamma\}I_1(\alpha pr) - \tfrac{1}{2}\sum_{m=1}^{\infty} \frac{(\frac{1}{2}\alpha pr)^{2m+1}}{m!\ (m+1)!}\left[\sum_{\frac{1}{\alpha}=1}^{m+1}\alpha + \sum_{\frac{1}{\alpha}=1}^{m}\alpha\right].$$

Restricting to the kind of approximation referred to above, we can write

$$I_1(\alpha pr) \sim \tfrac{1}{2}(\alpha pr)$$

$$K_1(\alpha pr) \sim a_1(\alpha pr) + \frac{b_1}{(\alpha pr)}$$

where a_1 and b_1 are numerical quantities.
Thus

$$\int_{r_1}^{r_2} \frac{I_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}\alpha p(r_2-r_1)$$

$$\int_{r_1}^{r_2} \frac{K_1(\alpha pr)}{r}\,dr \sim a_1\alpha p(r_2-r_1) + \frac{b_1}{\alpha p}\left(\frac{1}{r_1}-\frac{1}{r_2}\right).$$

So

$$I_1(\alpha pr_2)\int_{r_1}^{r_2} \frac{K_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}r_2 b_1\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

$$K_1(\alpha pr_2)\int_{r_1}^{r_2} \frac{I_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}\frac{b_1}{r_2}(r_2-r_1)$$

$$K_1(\alpha pr_2)I_1(\alpha pr_1) - I_1(\alpha pr_2)K_1(\alpha pr_1) \sim \tfrac{1}{2}b_1\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)$$

$$I_1(\alpha pr_1)\int_{r_1}^{r_2} \frac{K_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}a_1\alpha^2p^2r_1r_2 - \tfrac{1}{2}a_1(\alpha pr_1)^2 + \frac{b_1}{2} - \frac{b_1r_1}{2r_2}$$

$$K_1(\alpha pr_1)\int_{r_1}^{r_2} \frac{I_1(\alpha pr)}{r}\,dr \sim \tfrac{1}{2}a_1p^2\alpha^2r_1r_2 - \tfrac{1}{2}a_1(p\alpha r_1)^2 + \frac{b_1r_2}{2r_1} - \frac{b_1}{2}.$$

Hence from eqn. (25) it follows that

$$(\bar{u})_{r=r_1} \sim \frac{M}{hp^3}\cdot\frac{\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)}{\left\{\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)+\frac{r_2-r_1}{r_2}-r_2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}$$

$$+\frac{(V_0+N)}{hp}\cdot\frac{\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)}{\left\{\left(\frac{r_1}{r_2}-\frac{r_2}{r_1}\right)+\frac{r_2-r_1}{r_2}-r_2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}$$

$$+\frac{L}{p^3}\cdot\frac{\left[2-\left(\frac{r_2}{r_1}-\frac{r_1}{r_2}\right)\right]}{2\left(1-\frac{r_2}{r_1}\right)}+\frac{W}{p^3}$$

or

$$(\bar{u})_{r=r_1}\sim\frac{M}{2hp^3}\left(1+\frac{r_1}{r_2}\right)+\frac{(V_0+N)}{2hp}\left(1+\frac{r_1}{r_2}\right)+\frac{L(2r_1r_2-r_2^2-r_1^2)}{2r_2(r_1-r_2)p^3}+\frac{W}{p^3}.$$

Taking the inverse transform, the mechanical displacement of the inner boundary of the disk $r=r_1$, for large values of time, is given by

$$(u)_{r=r_1}\sim\frac{Mt^2}{4h}\left(1+\frac{r_1}{r_2}\right)+\frac{L(2r_1r_2-r_2^2-r_1^2)t^2}{4r_2(r_1-r_2)}+\frac{Wt^2}{2}$$

$$+\left[\frac{(V_0+N)}{2h}\left(1+\frac{r_1}{r_2}\right)\right]H(t).$$

Thus to conclude we say that the mechanical response owing to a step voltage input under thermal conditions is partly step in character, for large values of time. The similar result can also be had from (24). In the absence of temperature the result agrees with some known results.

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References

Cady, W. G. (1946). Piezoelectricity. McGraw-Hill Book Co. Inc., New York.

Carslaw, H. S., and Jaeger, J. C. (1959). Conduction of Heat in Solids, 2nd ed., Clarendon Press, Oxford.

Erdeyli, A. (1953). Higher Transcendental Functions, 2. McGraw-Hill Book Co. Inc., New York.

Giri, R. R. (1966). Note on the study of responses in a piezoelectric plate transducer when the input is partly periodic and partly periodic transient. *Revue roum. Sci. Techn.*, **11**, 253.

Mason, W. P. (1950). Piezoelectric Crystals and their Applications to Ultrasonics. D. Van Nostrand Co. Inc., New York.

Redwood, M. (1961*a*). Transient performance of a piezoelectric transducer. *J. acoust. Soc. Am.*, **33**, 527.

——— (1961*b*). Piezoelectric generation of an electrical impulse. *J. acoust. Soc. Am.*, **33**, 1386.

Sinha, D. K. (1962). Note on responses in a piezoelectric plate transducer with a periodic step input. *Indian J. theor. Phys.*, **10**, 21.

——— (1965). A note on mechanical responses in a piezoelectric transducer owing to an impulsive voltage input. *Proc. natn. Inst. Sci. India*, **31**, 395.

MAGNETO-THERMO-ELASTIC INTERACTIONS IN AN INFINITE SOLID DUE TO TRANSIENT HEAT SOURCES

by SACHINDRA KUMAR BAKSHI, *Department of Mathematics, Ramakrishna Mission Vidyamandira, Belur Math, Howrah*

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The object of the present paper is to determine the distribution of stress, strain, temperature, electric and magnetic fields in an infinite solid if it is subjected to a transient heat source. The solutions are first obtained in terms of Fourier and Laplace transforms. The inverse transforms are studied in particular cases.

INTRODUCTION

The studies in magneto-elasticity which has largely been possible due to the merger of the theories of electro-magnetism and elasticity are of recent origin. Apart from these, attempts are also being made in recent times to accommodate the thermal field in the interplay of electro-magnetic and mechanical fields. These studies are considered to be important primarily because of their diverse applications in various branches of geophysics, optics and in the devices such as aero-magnetic flutter and also in plasmatrons, plasmaguides, etc. Though there exists in abundance a literature of magneto-elasticity, vide Kaliski (1960*a*, *b* and 1961), Sinha (1965, 1966, 1967), Paria (1961), Giri (1966), Das (1967), the subject of magneto-thermo-elasticity is still in its formative period. The initial attempts in this direction have been constituted by Nowacki and Kaliski (1963), Nowacki (1962, 1963), Paria (1962, 1964), Wilson (1966). The papers of Sinha (1964) and Murthy (1966) should also be mentioned in this connection. The present paper is a similar attempt aiming at the determination of the effects of thermal and electro-magnetic conditions on the distribution of temperature, stresses as well as the amount of the induced magnetic field. The solution is achieved by making use of the equations of Maxwell, equation of elasticity and equation of heat conduction. It is found that the use of Fourier-Laplace transforms facilitates the solution of the problem.

FORMULATION OF THE PROBLEM, FUNDAMENTAL EQUATIONS

Let us consider an infinite elastic solid body in which an initial magnetic field is impressed and having a uniform temperature. There is no initial

stress or strain. The body is subjected to a time-decaying heat source placed on a plane extending in all directions. The plane is taken as y-z plane, z-axis being along the original magnetic field H_3. The x-axis is perpendicular to this plane. The displacement $\vec{u}$ has the components (u, o, o), u is a function of x and t.

The problem being one of magneto-thermo-elasticity, we have to solve Maxwell's electromagnetic equations, Fourier equation of heat conduction together with the equation of elasticity given by Hooke-Duhamel.

Maxwell's equations reduce to

$$\left.\begin{aligned} &\operatorname{curl} \vec{H} = \vec{j}, \quad \operatorname{div} \vec{B} = 0 \\ &\operatorname{curl} \vec{E} = -\frac{\partial B}{\partial t}, \quad \vec{B} = \mu_e \vec{H} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (1)$$

in absence of displacement current.

These equations should be supplemented by Ohm's law given by

$$\vec{j} = \sigma\left[\vec{E} + \frac{\partial \vec{u}}{\partial t} \times \vec{B}\right] - k_0 \operatorname{grad} T \quad .. \quad .. \quad .. \quad (2)$$

The Hooke-Duhamel law can be written as

$$\tau_{ij} = 2\mu e_{ij} + (\lambda e - \beta T)\delta_{ij} \quad .. \quad .. \quad .. \quad .. \quad (3)$$

where

$$\left.\begin{aligned} &e_{ij} = \tfrac{1}{2}(u_{i,\,j} + u_{j,\,i}), \quad e = \operatorname{div} \vec{u} \\ &\delta_{ij} = 1 \quad \text{for } i = j \text{ and } \delta_{ij} = 0 \quad \text{for } i \neq j \end{aligned}\right\} \quad .. \quad .. \quad (4)$$

The Fourier law of heat conduction is

$$k \nabla^2 T + Q = \rho c_v \frac{\partial T}{\partial t} + T_0 \beta \frac{\partial e}{\partial t} + \pi_0 \operatorname{div} \vec{j} \quad .. \quad .. \quad (5)$$

The equation of motion of the solid is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ik}}{\partial x_k} + \left(\vec{j} \times \vec{B}\right)_i \quad .. \quad .. \quad .. \quad .. \quad (6)$$

The symbols used here have their usual meanings as in Paria (1962). Q represents the intensity of the heat source, T_0 the initial uniform temperature of the solid, and T the perturbed temperature over and above T_0.

From eqns. (3), (4) and (6) we get

$$\rho \frac{\partial^2 \vec{u}}{\partial t^2} = \mu \nabla^2 \vec{u} + (\lambda + \mu) \operatorname{grad} e - \beta \operatorname{grad} T + \left(\vec{j} \times \vec{B}\right) \quad .. \quad .. \quad (7)$$

Solution of the Problem

We can assume for this problem from symmetry that all the physical quantities must be functions of x and t.

Let us take $Q = q_0\delta(x)e^{-\Omega t}(\Omega > 0)$ where $\delta(x)$ is the Dirac-delta function and q_0 the strength of the source. We get the following equations as given by Paria (1964):

$$E_x = \frac{k_0}{\sigma}\cdot\frac{\partial T}{\partial x}, \quad H_x = 0, \quad j_x = 0 \quad .. \quad .. \quad .. \quad (8)$$

$$\frac{\partial E_y}{\partial x} = -\mu_e\frac{\partial H_z}{\partial t}, \quad j_y = -\frac{\partial H_z}{\partial x}, \quad H_y = 0 \quad .. \quad .. \quad (9)$$

$$E_z = 0, \quad j_z = 0 \quad .. \quad .. \quad .. \quad .. \quad (10)$$

$$\frac{\partial h_z}{\partial t} = \nu_H\frac{\partial^2 h_z}{\partial x^2} - H_3\frac{\partial^2 u}{\partial x\partial t} \quad .. \quad .. \quad .. \quad .. \quad (11)$$

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial T}{\partial x} - \mu_e H_3\frac{\partial h_z}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \quad .. \quad .. \quad .. \quad (12)$$

$$k\frac{\partial^2 T}{\partial x^2} + q_0\delta(x)e^{-\Omega t} = \rho c_v\frac{\partial T}{\partial t} + T_0\beta\frac{\partial^2 u}{\partial x\partial t} \quad .. \quad .. \quad .. \quad (13)$$

where $\nu_H = \dfrac{1}{\mu_e\sigma}$ is the magnetic viscosity and $H_z = H_3 + h_z$. (h_z is so small that its square and higher powers and its product with u can be neglected).

Equations (11), (12) and (13) will determine h_z and u, T and hence from eqn. (8) we get E_x and from third equation of (9), j_y can be determined.

We now proceed to obtain the solution of the problem by means of transforms.

Let us define the Laplace and Fourier transforms of a function $f(x, t)$ by $\bar{f}$ and $\bar{f}'$, respectively where

$$\bar{f} = \int_0^\infty f(x, t)\exp(-pt)dt, \quad \bar{f}' = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}.\exp(i\xi x)dx. \ (\mathrm{Re}\ p > 0)$$

From equations (11), (12) and (13) we get

$$(p+\nu_H\xi^2)\bar{h}_z' = H_3 ip\xi\bar{u}' \quad .. \quad .. \quad .. \quad (14)$$

$$\mu_e H_3 i\xi\bar{h}_z' = \{\rho p^2 + (\lambda+2\mu)\xi^2\}\bar{u}' - \beta i\xi\bar{T}' \quad .. \quad .. \quad (15)$$

$$\frac{q_0}{\sqrt{2\pi}} = \{\rho c_v p(p+\Omega) + (p+\Omega)k\xi^2\}\bar{T}' - T_0\beta i\xi p(p+\Omega)\bar{u}' \quad .. \quad .. \quad (16)$$

Solving these equations we get

$$\bar{T}' = \frac{q_0}{\sqrt{2\pi}}\cdot\frac{L_1}{L_2}$$

$$\bar{u}' = \frac{q_0}{\sqrt{2\pi}}\cdot\frac{\beta i\xi(p+\nu_H\xi^2)}{L_2}$$

$$\bar{h}_z' = -\frac{q_0 H_3 p\beta\xi^2}{\sqrt{2\pi}L_2}$$

where

$$L_1 = \nu_H(\lambda+2\mu)\xi^4+p\left(\nu_H\rho p+\lambda+2\mu+\mu_e H_3^2\right)\xi^2+\rho p^3$$

$$L_2 = \{\rho c_v p(p+\Omega)+(p+\Omega)k\xi^2\}L_1+T_0\beta^2\xi^2 p(p+\Omega)(p+\nu_H\xi^2)$$

Putting

$$\left.\begin{aligned} R_H &= \frac{\mu_e H_3^2}{\rho c_1^2}, \quad \epsilon_T = \frac{T_0\beta^2}{\rho^2 c_v \cdot c_1^2} \\ \text{and} \qquad k_1 &= \frac{k}{\rho c_v} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (17)$$

The above relations can be put in the form

$$\bar{T}' = \frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{M_1}{M_2} \quad .. \quad .. \quad .. \quad .. \quad (18)$$

$$\bar{u}' = \frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{\beta i\xi(p+\nu_H\xi^2)}{\rho M_2} \quad .. \quad .. \quad .. \quad (19)$$

$$h_z' = -\frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{H_3 p\beta\xi^2}{\rho M_2} \quad .. \quad .. \quad .. \quad .. \quad (20)$$

where

$$\left.\begin{aligned} M_1 &= \left(\nu_H c_1^2\xi^2+\xi_1^2\right)\left(\xi^2+\xi_1^2\right) \\ \xi_1 &= \tfrac{1}{2}p\left(\nu_H p+c_3^2+\sqrt{N}\right) \\ \nu_H c_1^2\xi_2^2 &= \tfrac{1}{2}p\left(\nu_H p+c_3^2-\sqrt{N}\right) \\ N &= \nu_H p^2-2\nu_H p\left(2c_1^2-c_3^2\right)+c_3^4 \\ c_1^2 &= \frac{\lambda+2\mu}{\rho}, \quad c_3^2 = c_1^2(1+R_H) \\ \text{and} \qquad M_2 &= (p+k_1\xi^2)M_1+c_1^2 p\xi^2(p+\nu_H\xi^2)\epsilon_T \end{aligned}\right\} \quad .. \quad .. \quad (21)$$

Using the second and third equations of (9) and the first equation of (8) and taking transforms, we get

$$\left.\begin{aligned} \bar{j}_y' &= -\frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{H_3 p\beta i\xi^3}{\rho M_2} \\ \bar{E}_y' &= \frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{\mu_e H_3\beta i\xi p}{\rho M_2} \\ \bar{E}_x' &= -\frac{K_0}{\sigma}\cdot\frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{i\xi M_1}{M_2} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (22)$$

By taking the inverse transforms in eqns. (18)–(20) and (22) we can obtain T, u, h_z, E_x, E_y, j_y. The other components of $\vec{H}$, $\vec{E}$ and $\vec{j}$ have already

been proved to be zero. The only non-vanishing stress component is τ_{xx} which can be derived from eqn. (3) in the form

$$\tau_{xx} = (\lambda+2\mu)\frac{\partial u}{\partial x} - \beta T$$

Taking the transforms we obtain

$$\bar{\tau}'_{xx} = -\frac{q_0}{\rho c_v\sqrt{2\pi}}\cdot\frac{\beta p}{M_2}\cdot\{\xi^2(\nu_H p + c_1^2 R_H)+p^2\} \quad .. \quad .. \quad (23)$$

For a perfect conductor we have $\sigma \to \infty$ so that $\nu_H = 0$. Therefore,

$$\left.\begin{aligned} M_1 &= p(p+\Omega)\left[c_3^2 k_1\xi^4 + p\xi^2(c_5^2+k_1 p)+p^3\right] \\ M_2 &= p(p+\Omega)(c_3^2 k_1\xi^2+\xi_3^2)(\xi^2+\xi_4^2) \end{aligned}\right\} \quad .. \quad .. \quad (24)$$

where

$$c_5^2 = c_3^2 + c_1^2\epsilon_T = c_1^2(1+R_H+\epsilon_T) \quad .. \quad .. \quad .. \quad .. \quad (25)$$

$$\xi_3^2 = \tfrac{1}{2}p(k_1 p + c_5^2 + \sqrt{R}) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (26)$$

$$\xi_4^2 = \frac{p}{2k_1 c_3^2}(k_1 p + c_5^2 - \sqrt{R}) \quad .. \quad .. \quad .. \quad .. \quad (27)$$

$$R = k_1^2 p^2 - 2k_1 p(2c_3^2 - c_5^2) + c_5^4 \quad .. \quad .. \quad .. \quad .. \quad (28)$$

Therefore, it follows from eqn. (18) that

$$\bar{T}' = \frac{q_0}{\rho c_v\sqrt{2\pi}\,.\,(p+\Omega)}\left(\frac{A}{c_3^2 k_1\xi^2+\xi_3^2} + \frac{B}{\xi^2+\xi_4^2}\right) \quad .. \quad .. \quad (29)$$

where

$$A = \tfrac{1}{2}c_3^2\left(1 - \frac{k_1 p - c_5^2}{\sqrt{R}}\right) \quad .. \quad .. \quad .. \quad .. \quad (30)$$

$$B = \frac{1}{2k_1}\left(1 + \frac{k_1 p - c_5^2}{\sqrt{R}}\right) \quad .. \quad .. \quad .. \quad (31)$$

Taking the inverse Fourier transform of $\bar{T}'$, we get

$$\bar{T} = \frac{q_0}{2\rho c_v p^{\frac{1}{2}}(p+\Omega)}\left[\frac{A}{c_3\sqrt{k_1}}\,.\,\xi_4\,.\exp\left(\frac{-\xi_3 x}{c_3\sqrt{k_1}}\right) + B\xi_3\exp(-\xi_4 x)\right] \quad .. \quad (32)$$

Similarly from eqn. (20) we get

$$\bar{h}'_z = -\frac{q_0 H_3\beta}{\rho^2 c_v\sqrt{2\pi}\,.\,p(p+\Omega)}\cdot\frac{1}{\sqrt{R}}\left(\frac{\xi_3^2}{c_3^2 k_1\xi^2+\xi_3^2} - \frac{\xi_4^2}{\xi^2+\xi_4^2}\right)$$

The inverse Fourier transform then gives

$$\bar{h}_z = -\frac{q_0 H_3\beta}{2\rho^2 p(p+\Omega)\sqrt{R}}\left[\frac{\xi_3}{c_3\sqrt{k_1}}\exp\left(\frac{-\xi_3 x}{c_3\sqrt{k_1}}\right) - \xi_4\exp(-\xi_4 x)\right] \quad (33)$$

From eqn. (23) we get $\bar{\tau}'_{xx}$. Putting $\nu_H = 0$ and then taking the inverse Fourier transform we have

$$\bar{\tau}_{xx} = -\frac{q_0\beta}{2\rho c_v p^{\frac{3}{2}}(p+\Omega)}\left[\frac{A_1\xi_4}{c_3\sqrt{k_1}}\exp\left(\frac{-\xi_3 x}{c_3\sqrt{k_1}}\right)+B_1\xi_3\exp\left(-\xi_4 x\right)\right] \quad .. \quad (34)$$

where

$$A_1 = \frac{c_1^2}{p\sqrt{R}}\left[R_H\xi_3^2-(1+R_H)p^2k_1\right] \quad .. \quad .. \quad .. \quad (35)$$

$$B_1 = -\frac{1}{p\sqrt{R}}(c_1^2R_H\xi_4^2-p^2) \quad .. \quad .. \quad .. \quad (36)$$

Temperature Field for Small Thermo-elastic Coupling

ϵ_T is called the thermo-elastic coupling factor which is very small for common metals. We shall expand various expressions containing ϵ_T retaining only its first power.

From eqn. (28) we have

$$\sqrt{R} = (k_1p-c_3^2)+\frac{k_1p+c_3^2}{k_1p-c_3^2}\cdot c_1^2\epsilon_T \quad .. \quad .. \quad .. \quad (37)$$

Hence from eqns. (26) and (27) we get

$$\xi_3 = p\sqrt{k_1}\left[1+\frac{c_1^2\epsilon_T}{2(k_1p-c_3^2)}\right] \quad .. \quad .. \quad .. \quad (38)$$

$$\xi_4 = \sqrt{\frac{p}{k_1}}\left[1-\frac{c_1^2\epsilon_T}{2(k_1p-c_3^2)}\right] \quad .. \quad .. \quad .. \quad (39)$$

Also from eqns. (30) and (31) we obtain

$$A = \frac{c_1^2c_3^2\epsilon_T\cdot k_1p}{(k_1p-c_3^2)^2} \quad .. \quad .. \quad .. \quad .. \quad (40)$$

$$B = \frac{1}{k_1}\left[1-\frac{c_1^2k_1p\epsilon_T}{(k_1p-c_3^2)^2}\right] \quad .. \quad .. \quad .. \quad (41)$$

With these approximated values, we have from eqn. (32)

$$\frac{2\rho c_v\bar{T}}{q_0} = \frac{1}{\sqrt{k_1p}\cdot(p+\Omega)}\cdot e^{-x\sqrt{\frac{p}{k_1}}}+\frac{c_1^2\epsilon_T}{p+\Omega}\left[\frac{c_3e^{-\frac{px}{c_3}}}{(k_1p-c_3^2)^2}+\frac{x}{2k_1(k_1p-c_3^2)}\cdot e^{-x\sqrt{\frac{p}{k_1}}}\right.$$

$$\left.-\frac{1}{4}\left\{\frac{1}{(\sqrt{k_1p}-c_3)^2}+\frac{1}{(\sqrt{k_1p}+c_3)^2}\right\}\times\frac{1}{\sqrt{k_1p}}e^{-x\sqrt{\frac{p}{k_1}}}\right] \quad .. \quad .. \quad (42)$$

Taking the inverse Laplace transform, vide Erdelyi (1954), and introducing the dimensionless quantity $\eta = \frac{xc_1}{k_1}$, we obtain

$$\frac{2\rho c_v k_1}{q_0 c_1} \cdot T = e^{-\Omega t} \cdot \frac{2k_1}{c_1^2\sqrt{\pi}} \cdot \int_0^{\frac{c_1}{\sqrt{k_1}}\sqrt{t}} e^{-\frac{\eta^2}{4\tau^2} + \frac{\Omega k_1}{c_1^2}\tau^2} d\tau + \epsilon_T F(\eta, t, R_M) \quad .. \quad (43)$$

where

$$R_M = 1 + R_H \quad .. \quad .. \quad .. \quad .. \quad (44)$$

and

$$F(\eta, t, R_M) = -\frac{e^{-\Omega t}}{\sqrt{\pi}} \cdot \frac{2k_1}{c_1^2} \cdot \int_0^{\frac{c_1}{\sqrt{k_1}}\sqrt{t}} \tau^2 \cdot e^{\left(\frac{\Omega k_1}{c_1^2}\tau - \frac{\eta^2}{4\tau^2}\right)} d\tau$$

$$+\frac{k_1}{c_1^2} e^{-\Omega t} \int_0^{\frac{c_1}{\sqrt{k_1}}\sqrt{t}} \left[\tau \cdot e^{\frac{\Omega k_1}{c_1^2}\tau} \left\{(\eta - \tau^2\sqrt{R_M}) e^{\tau^2 R_M - \eta\sqrt{R_M}}\right.\right.$$

$$\times \operatorname{erfc}\left(\frac{\eta}{2\tau} - \tau\sqrt{R_M}\right) + (\eta + \tau^2\sqrt{R_M}) \cdot e^{\tau^2 R + \eta M\sqrt{R_M}}$$

$$\left.\left. \times \operatorname{erfc}\left(\frac{\eta}{2\tau} + \tau\sqrt{R_M}\right)\right\}\right] d\tau \quad \text{for } t < \frac{x}{c_3} = \frac{\eta k_1}{c_1 c_3} \quad .. \quad .. \quad (45)$$

and

$$= \frac{c_1 c_3}{k_1}\left[\frac{e^{-\Omega\left(t - \frac{\eta k_1}{c_1 c_3}\right)}}{\left(\Omega + \frac{c_3^2}{k_1}\right)^2} - \frac{1}{\left(\Omega + \frac{c_3^2}{k_1}\right)} \cdot e^{\frac{c_3^2}{k_1}\left(t - \frac{\eta k_1}{c_1 c_3}\right)} + \frac{1}{\left(\Omega - \frac{c_3^2}{k_1}\right)} \cdot \left(t - \frac{\eta k_1}{c_1 c_3}\right) \cdot e^{\frac{c_3^2}{k_1}\left(t - \frac{\eta k_1}{c_1 c_3}\right)}\right]$$

$$-\frac{e^{-\Omega t}}{\sqrt{\pi}} \cdot \frac{2k_1}{c_1^2} \int_{\frac{\eta k_1}{c_1 c_3}}^{\frac{c_1}{\sqrt{k_1}}\sqrt{t}} \tau^2 \cdot e^{\left(\frac{\Omega k_1}{c_1^2}\tau - \frac{\eta^2}{4\tau^2}\right)} \cdot d\tau$$

$$+\frac{k_1}{c_1^2} e^{-\Omega t} \int_{\frac{\eta k_1}{c_1 c_3}}^{\frac{c_1}{\sqrt{k_1}}\sqrt{t}} \left[\tau \cdot e^{\frac{\Omega k_1 \tau}{c_1^2}} \left\{(\eta - \tau^2\sqrt{R_M}) e^{\tau^2 R_M - \eta\sqrt{R_M}} \cdot \operatorname{erfc}\left(\frac{\eta}{2\tau} - \tau\sqrt{R_M}\right)\right.\right.$$

$$\left.\left. + (\eta + \tau^2\sqrt{R_M}) e^{\tau^2 R_M + \eta\sqrt{R_M}} \cdot \operatorname{erfc}\left(\frac{\eta}{2\tau} + \tau\sqrt{R_M}\right)\right\}\right] \quad \text{for } t > \frac{\eta k_1}{c_1 c_3} \quad .. \quad .. \quad (46)$$

From eqn. (43), it follows that the first term on the right-hand side which decays with time exponentially represents the solution of the classical heat conduction equation while the function $F(\eta, t, R_M)$ is the perturbation due to the thermo-elastic coupling factor ϵ_T. The initial magnetic field R_M occurs only in the perturbed function.

Table I gives the values of $F(\eta, t, R_M)$ for different values of η, R_M when $\Omega t = 0{\cdot}72$, the conductor being aluminium.

TABLE I

Values of $F(\eta, t, R_M)$ in units of 10^{-11}

η / R_M	0·0	0·6	1·2	1·8	2·4	3·0	3·6
1·0	−3·487	−1·043	−0·179	0·016	0·015	0·004	0·003
2·0	−8·278	−2·526	−0·648	−0·089	−0·002	0·012	0·015

The continuous curve as in Fig. 1 represents the perturbation when the magnetic field is absent while the dotted curve the perturbation function when the magnetic field parameter $R_M = 2$. It is observed from the graph that the magnetic field reduces the temperature near the end ($\eta = 0$) of the solid while, in the rest of the region, temperature is not much affected by the presence of the magnetic field.

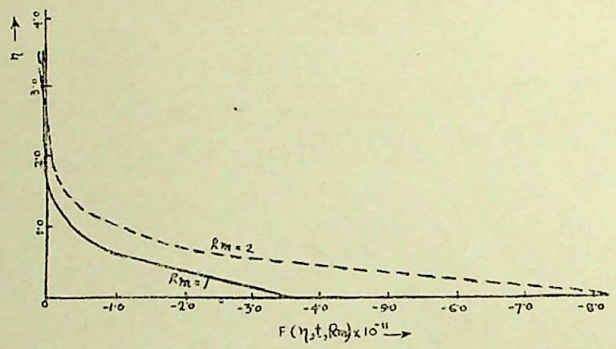

FIG. 1. Effect of magnetic field parameter on perturbation due to thermo-elastic coupling factor.

DISTRIBUTION OF STRESS AND INDUCED MAGNETIC FIELD FOR SMALL TIME

The stress and induced magnetic field are determined considering small values of time. In this case p is very large. Therefore, we can take

$$\sqrt{R} = k_1 p, \quad \xi_3 = \sqrt{k_1}\,.\,p\left(1+\frac{c_5^2}{4k_1 p}\right)$$

$$\xi_4 = \frac{c_5}{c_3}\sqrt{\frac{p}{2k_1}}$$

in place of relations (26), (27) and (28) respectively.

Relations (35) and (36) reduce to

$$A_1 = -c_1^2\left(1-\frac{c_5^2 R_M}{2k_1 p}\right), \quad B_1 = \frac{1}{k_1}\left(1-\frac{c_1^2 c_5^2 R_M}{2c_3^2 k_1 p}\right)$$

Equation (33) reduces to

$$\bar{h}_z = -\frac{q_0 H_3 \beta}{2\rho^2 c_v k_1 p^2 (p+\Omega)\,.\,c_3}\left[p\left(1+\frac{c_5^2}{4k_1 p}\right)e^{-\frac{px}{c_3}\left(1+\frac{c_5^2}{4k_1 p}\right)}\right.$$

$$\left. -c_5\sqrt{\frac{p}{2k_1}}\,e^{-(c_5/c_3)}\,.\,\sqrt{\frac{p}{2k_1}}\,.\,x\right] \quad .. \quad .. \quad .. \quad .. \quad (47)$$

Taking the inverse Laplace transform, we get

$$h_z = -\frac{q_0 H_3 \beta}{2\rho^2 c_v k_1 c_3}\,\phi(\eta, t, R_M, \epsilon_T)$$

where

$$\eta = \frac{x c_1}{k_1}$$

and

$$\phi(\eta, t, R_M, \epsilon_T)$$

$$= -\frac{2k_1}{c_1^2}(\sqrt{R_M+\epsilon_T})\,.\,e^{-\Omega t}\int_0^{c_1\sqrt{\frac{t}{k_1}}}\left[\sqrt{\frac{2}{\pi}}\,.\,\tau^2\,.\,\exp\left\{\frac{\Omega k_1}{c_1^2}\tau^2-\frac{\eta^2}{\tau^2}\left(1+\frac{\epsilon_T}{R_M}\right)\right\}\right.$$

$$\left. -\frac{\eta\tau}{2}\left(\sqrt{1+\frac{\epsilon_T}{R_M}}\right).\,e^{\frac{\Omega k_1}{c_1^2}\tau^2}\,\mathrm{erfc}\left(\frac{\eta}{2\sqrt{2}\tau}\,.\,\sqrt{1+\frac{\epsilon_T}{R_M}}\right)\right]d\tau \text{ for } t<\frac{\eta k_1}{c_1 c_3} \quad .. \quad (48)$$

and

$$= \frac{1}{\Omega}\left\{1-e^{-\Omega t+\frac{\eta k_1 \Omega}{c_1 c_3}}\right\}\,.\,e^{-\frac{\eta}{4\sqrt{R_m}}\,.\,(R_M+\epsilon_T)}+\frac{c_1^2}{4k_1\Omega}(R_M+\epsilon_T)\,.\,e^{-\frac{\eta}{4\sqrt{R_m}}\,.\,(R_M+\epsilon_T)}$$

$$\times\left\{t-\frac{\eta k_1}{c_1 c_3}+\frac{1}{\Omega}e^{-\Omega t+\frac{\eta k_1\Omega}{c_1 c_3}}-\frac{1}{\Omega}\right\}$$

$$-\frac{2k_1}{c_1^2}\sqrt{R_M+\epsilon_T}\,.\,e^{-\Omega t}\,.\int_{\frac{\eta k_1}{c_1 c_3}}^{c_1\sqrt{\frac{t}{k_1}}}\left[\sqrt{\frac{2}{\pi}}\,.\,\tau^2\exp\left\{\frac{\Omega k_1}{c_1^2}\tau^2-\frac{\eta^2}{\tau^2}\left(1+\frac{\epsilon_T}{R_M}\right)\right\}\right.$$

$$\left. -\frac{\eta\tau}{2}\sqrt{1+\frac{\epsilon_T}{R_M}}\,.\,e^{\frac{\Omega k_1\tau^2}{c_1^2}}\,\mathrm{erfc}\left(\frac{\eta}{2\sqrt{2}\tau}\,.\,\sqrt{1+\frac{\epsilon_T}{R_M}}\right)\right]d\tau \text{ for } t>\frac{\eta k_1}{c_1 c_3} \quad .. \quad .. \quad (49)$$

To study the effect of the magnetic field parameter on the induced field we can put $\epsilon_T = 0$ since ϵ_T occurs with R_M which is very large in comparison with ϵ_T.

Table II gives the values of $\phi(\eta)$ when $\Omega t = 0{\cdot}72$, $\epsilon_T = 0$, $R_M = 1{\cdot}5$.

TABLE II

Values of $\phi(\eta)$ in units of 10^{-12}

η	0	·1	·2	·3	·4	·5	·6
$\phi(\eta)$	1·043	0·797	0·904	0·911	0·943	0·944	1·060

From the graph (Fig. 2) it is easily seen that the effect of R_M at a certain instant of time gradually increases along the length of the solid beginning from a particular point of its length.

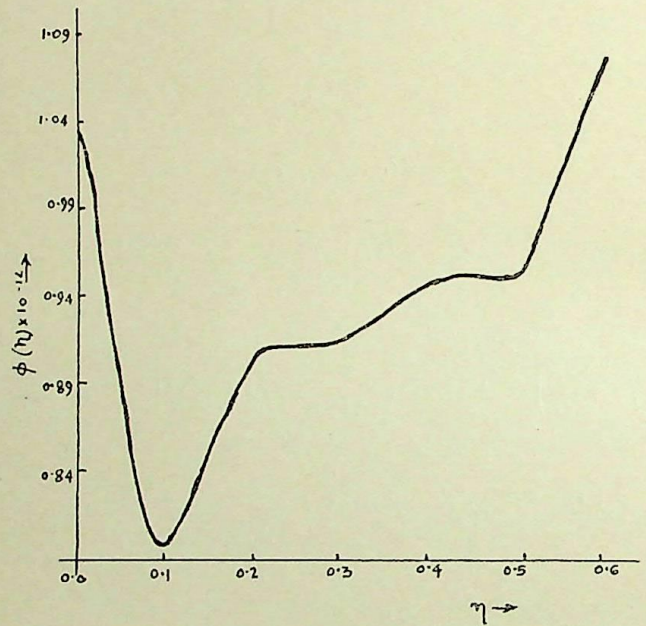

FIG. 2. Effect of magnetic field parameter on induced magnetic field.

Taking into consideration the approximations for small-time, equation (34) reduces to

$$\bar{\tau}_{xx} = \frac{q_0\beta}{2\rho c_v(p+\Omega)}\left[\frac{c_1^2 c_5}{\sqrt{2}c_3^2 k_1 p}\left(1-\tfrac{1}{2}R_H\cdot\frac{c_5^2}{k_1 p}\right)e^{-\frac{px}{c_3}-\frac{c_5^2 x}{4k_1 c_3}}\right.$$

$$\left.-\frac{1}{\sqrt{k_1 p}}\left\{1+\frac{c_5^2(c_3^2-2R_H c_1^2)}{4k_1 c_3^2 p}\right\}e^{-\frac{c_5}{c_3}x\cdot\sqrt{\frac{p}{2k_1}}}\right] \quad \ldots \quad \ldots \quad (50)$$

The inverse Laplace transform then gives

$$\tau_{xx} = \frac{q_0\beta}{2\rho c_v}\psi(\eta, t, R_M, \epsilon_T)$$

where

$$\psi(\eta, t, R_M, \epsilon_T) = \frac{2}{\sqrt{\pi}\,.\,c_1}\int_0^{c_1\sqrt{\frac{t}{k_1}}} .\, e^{-\Omega t+\frac{\Omega k_1}{c_1^2}\tau-\frac{R_M+\epsilon_T}{8}\cdot\frac{\eta^2}{\tau^2}}\,.\,d\tau$$

$$-\frac{1}{\sqrt{k_1}}\cdot\frac{R_M+\epsilon_T}{2R_M}\,.\,(2-R_M)\times\int_0^{c_1\sqrt{\frac{t}{k_1}}} e^{-\Omega t+\frac{\Omega k_1}{c_1^2}\tau}\,.\,\tau\left\{\frac{2\sqrt{k_1\tau}}{c_1\sqrt{\pi}}\right.$$

$$\left.\times e^{-\frac{R_M+\epsilon_T}{8}\cdot\frac{\eta^2}{\tau^2}}-\sqrt{\frac{R_M+\epsilon_T}{2R_M}}\,.\,\eta\,.\,\frac{\sqrt{k_1}}{c_1}\,\mathrm{erfc}\left(\sqrt{\frac{R_M+\epsilon_T}{2R_M}}\cdot\frac{\eta}{2\tau}\right)\right\}d\tau$$

$$\text{for } t<\frac{\eta k_1}{c_1c_3}$$

and

$$=\sqrt{\frac{R_M+\epsilon_T}{2}}\cdot\frac{c_1}{R_Mk_1\Omega}\cdot e^{-\frac{R_M+\epsilon_T}{4\sqrt{R_M}}\eta}\,.\left\{1-e^{-\Omega t+\frac{\Omega k_1\eta}{c_1^2\sqrt{k_M}}}\right\}$$

$$-\frac{R_M-1}{2\sqrt{2}R_M}\cdot\frac{c_1^3\,.\,(R_M+\epsilon_T)^{\frac{3}{2}}}{k_1^2}\cdot\left\{\frac{1}{\Omega}\left(t-\frac{\Omega k_1\eta}{c_1^2\sqrt{R_M}}\right)\right.$$

$$\left.+\frac{1}{\Omega^2}\cdot e^{-\Omega t+\frac{\Omega k_1}{c_1^2}}\cdot\frac{\eta}{\sqrt{R_M}}-\frac{1}{\Omega^2}\right\}\,.\,e^{-\frac{R_M+\epsilon_T}{4\sqrt{R_M}}\,.\,\eta}$$

$$+\frac{2}{\sqrt{\pi}c_1}\int_{\frac{\eta k_1}{c_1c_3}}^{c_1\sqrt{\frac{t}{k_1}}} e^{-\Omega t+\frac{\Omega k_1}{c_1^2}\tau-\frac{R_M+\epsilon_T}{8}\cdot\frac{\eta^2}{\tau^2}}\,.\,d\tau$$

$$-\frac{1}{\sqrt{k_1}}\cdot\frac{R_M+\epsilon_T}{R_M}\,.\,(2-R_M)\,.\int_{\frac{\eta k_1}{c_1c_3}}^{c_1\sqrt{\frac{t}{k_1}}} e^{-\Omega t+\frac{\Omega k_1}{c_1^2}\,.\,\tau}\,.\,\tau\left\{\frac{2\sqrt{k_1\tau}}{c_1\sqrt{\pi}}\right.$$

$$\left.\times e^{-\frac{R_M+\epsilon_T}{8}\cdot\frac{\eta^2}{\tau^2}}-\sqrt{\frac{R_M+\epsilon_T}{2R_M}}\,.\,\eta\,.\,\frac{\sqrt{k_1}}{c_1}\,.\,\mathrm{erfc}\left(\sqrt{\frac{R_M+\epsilon_T}{2R_M}}\,.\,\frac{\eta}{2\tau}\right)\right\}d\tau$$

$$\text{for } t>\frac{\eta k_1}{c_1c_3}$$

Here also the thermo-elastic coupling factor ϵ_T occurs with R_M. Hence to find the effect of magnetic field parameter we put $\epsilon_T = 0$.

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References

Das, N. C. (1967). A note on disturbances in an inhomogeneous elastic bar acted upon by a magnetic field. *Archwm. Mech. stosow.*, **19**, 765.

Erdelyi, A. (1954). Tables of Integral Transforms, Vol. I. McGraw-Hill Book Co. Inc., New York, p. 327.

Giri, R. R. (1966). Magneto-elastic disturbances in a transversely isotropic cylinder in a magnetic field. *Beitr. Geophysik*, **75**, 371.

Kaliski, S. (1960*a*). Dynamical equations of motion coupled with the field of temperatures and resolving functions for elastic and inelastic anisotropic bodies in the magnetic field. *Proc. Vibr. Probl.*, **1**, 4.

——— (1960*b*). Solution of the equations of notion for an infinite isotropic body in a magnetic field for a real conductor. *Archwm. Mech. stosow.*, **12**, 229.

——— (1961). The Propagation of elasto-plastic waves in a half-space in a magnetic field for a perfect conductor. Problems of Continuum Mechanics. Published by the Society for Industrial and Applied Mathematics, Pennsylvania, p. 229.

Kaliski, S., and Nowacki, W. (1963). Combined elastic and thermomagnetic waves produced by thermal shock in the case of a medium of finite thermal conductivity. *Int. J. Engng Sci.*, **1**, 163.

Murthy, S. N. (1966). Magneto-thermo-elastic interactions on the cooling process of an infinite circular cylinder. *J. Sci. Engng Res.*, **10**, 119.

Nowacki, W. (1962). Two dimensional problem of magneto-thermo-elasticity I. *Bull. Acad. pol. Sér. Sci. tech.*, **10**, 485.

——— (1963). The plane problem of Magneto-thermo-elasticity II. *Bull. Acad. pol. Sci. Sér. Sci. tech.*, **11**, 1.

Paria, G. (1961). Magneto-elastic stresses in a plane medium with a circular hole. *J. Sci. Engng Res.*, **5**, 41.

——— (1962). On magneto-thermo-elastic plane waves. *Proc. Camb. phil. Soc.*, 58, 627.

——— (1964). Magneto-thermo-elastic interactions in an infinite solid due to instantaneous heat sources. *Proc. Vibr. Probl.*, **5**, 57.

Sinha, D. K. (1964). On the Propagation of plane waves in a porous viscoelastic material containing a fluid and deformed by a magnetic field. *Geofis. pura appl.*, 59, 27.

——— (1965). A note on torsional disturbances in an elastic cylinder in a magnetic field. *Proc. Vibr. Probl.*, 6, 91.

——— (1966). Note on magneto-elastic disturbances in an elastic cylinder in a magnetic field. *Bull. Acad. pol. Sér. Sci. tech.*, **14**, 481.

——— (1967). A note on disturbances in a semi-infinite elastic half space placed in a magnetic field. *Beitr. Geophysik*, 76, 34.

Wilson, W. (1966). The propagation of magneto-elastic plane waves. *Proc. Camb. phil. Soc.*, 62, 91.

STABILITY OF CERTAIN TYPES OF FORCE-FREE PERIODIC FIELDS

by M. R. RAGHAVACHAR, *Department of Applied Mathematics, Indian Institute of Science, Bangalore 12*

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The stability of a certain type of periodic force-free fields having neutral lines, investigated by Schatzman on the basis of energy principle, is discussed. The normal mode technique is employed restricting to neighbourhood of neutral lines. The neutral lines contribute to instability in agreement with Schatzman. Repetition of stability analysis employing energy principle revealed that every cell appears to be stable contrary to Schatzman's conclusion that instabilities can be present under certain conditions. Moreover, the present study provides the growth rate of the instabilities.

1. INTRODUCTION

It was pointed out by Lüst and Schlüter (1954) that cosmic magnetic fields occurring in regions of low density should be force-free. As the ionized matter is highly conducting large currents can flow giving rise to large Lorentz force predominating over other forces such as arising from pressure gradient and gravitation or inertia. Consequently the Lorentz force should vanish to maintain steady states over considerably long time scales. Such types of fields have been characterized by Chandrasekhar (1956), Bhatnagar (1957) and Menzel (1964). In particular, Menzel has given a class of periodic force-free fields which could support quiescent prominences. Stability of the force-free fields has been investigated by Trehan (1957), Woltjer (1958) and Chakraborty and Bhatnagar (1960). Schatzman (1960) has discussed the stability of the periodic force-free fields of the type

$$\vec{H} = H_0\left[-\frac{l}{\alpha}\cos \kappa x \sin ly, \frac{\kappa}{\alpha}\sin \kappa x \cos ly, \cos \kappa x \cos ly\right] \quad .. \quad (1.1)$$

suggested by Menzel, where $\kappa^2+l^2 = \alpha^2$, α being the constant of proportionality of current to magnetic field. We note in particular that these fields have neutral lines. He has employed the energy principle given by Bernstein *et al.* (1958) and obtained certain necessary conditions for instability.

In the present study we discuss the stability of an ideally conducting homogeneous medium in presence of fields of the type (1.1) by normal mode technique, restricting ourselves to neighbourhood of neutral lines, as the

general discussion is mathematically very complicated. For the above fields the neutral lines are given by

$$x = \frac{\pi}{2\kappa}(2m+1), \quad y = \frac{\pi}{2l}(2n+1); \quad m; n = 0, \pm 1, \pm 2, \ldots \qquad \ldots \quad (1.2)$$

We find that neutral lines contribute to instability which agrees with the result obtained by Schatzman. But the conclusion drawn by him that there can be instabilities present along z-axis if $r^2 < \frac{\alpha^2 B^2}{32\pi} \frac{\sin^2 2\omega}{\gamma p_0}$ does not seem to be correct as can be verified by repeating the Schatzman stability analysis on the basis of energy principle. Contrary to Schatzman's conclusion we find that in this case there is always stability.

2. Linearized Hydromagnetic Equations

We shall work through the physical quantities rendered dimensionless according to the following scheme:

$$(x', y', z') = \alpha(x, y, z), \quad t' = (\alpha V_A)t$$

$$\delta\rho' = \frac{\delta\rho}{\rho_0}, \quad \delta p' = \frac{\delta p}{\gamma p_0}, \quad \vec{v}' = \frac{\vec{v}}{V_A}, \quad \vec{h}' = \frac{\vec{h}}{H_0}$$

where $V_A = \sqrt{\frac{H_0^2}{4\pi\rho_0}}$ is the Alfvén speed, p_0, ρ_0 being the pressure and density of the medium in equilibrium.

The linearized hydromagnetic equations in dimensionless form are:

$$\frac{\partial}{\partial t}(\delta\rho) + \operatorname{div} \vec{v} = 0 \qquad \ldots \quad (2.1)$$

$$\delta p = \delta\rho \qquad \ldots \quad (2.2)$$

$$\frac{\partial \vec{v}}{\partial t} = -S^2 \nabla(\delta p) + \vec{H}_0 \times \vec{h} + (\operatorname{curl} \vec{h}) \times \vec{H}_0 \qquad \ldots \quad (2.3)$$

$$\frac{\partial \vec{h}}{\partial t} = \operatorname{curl}(\vec{v} \times \vec{H}_0) \qquad \ldots \quad (2.4)$$

where $S^2 = (c_s^2/V_A^2)$, $\vec{H}_0$ is dimensionless steady state magnetic field, and $c_s = (\gamma p_0/\rho_0)^{\frac{1}{2}}$ is the sound velocity in the undisturbed medium.

3. Dispersion Relation

Taking perturbed quantities proportional to

$$\exp(i\omega t + ikz)$$

ω and k being dimensionless frequency and wave number respectively eqns. (2.1)–(2.4) reduce to after eliminating $\vec{h}$, δp and $\delta\rho$:

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$$(S^2+H_y^2+H_z^2)u_{xx}+H_y^2u_{yy}+2\left(H_z\frac{\partial H_z}{\partial x}+H_y\frac{\partial H_y}{\partial x}\right)u_x+\left[2ikH_yH_z-H_y\frac{\partial H_x}{\partial x}\right.$$

$$\left.+H_y\frac{\partial H_y}{\partial y}\right]u_y+\left[\omega^2-H_y\frac{\partial H_z}{\partial x}+H_z\frac{\partial H_y}{\partial x}+ikH_z\left(ikH_z-\frac{\partial H_x}{\partial x}\right)+H_z\frac{\partial^2 H_z}{\partial x^2}\right.$$

$$\left.+H_y\frac{\partial^2 H_y}{\partial x^2}+H_y\left(ik\frac{\partial H_z}{\partial y}-\frac{\partial^2 H_x}{\partial x\partial y}\right)\right]u-H_xH_yv_{xx}+(S^2+H_x^2)v_{xy}-H_xH_yv_{yy}$$

$$-\left[ikH_yH_z-H_y\frac{\partial H_y}{\partial y}+H_y\frac{\partial H_x}{\partial x}\right]v_x+\left[-H_yH_z-ikH_zH_x+H_z\frac{\partial H_z}{\partial x}-2H_y\frac{\partial H_x}{\partial y}\right]v_y$$

$$+\left[-H_y\frac{\partial H_z}{\partial y}-H_z\left(ikH_z-\frac{\partial H_y}{\partial y}\right)-ikH_z\frac{\partial H_x}{\partial y}+H_z\frac{\partial^2 H_z}{\partial x\partial y}-H_y\left(ik\frac{\partial H_z}{\partial x}-\frac{\partial^2 H_y}{\partial x\partial y}\right)\right.$$

$$\left.-H_y\frac{\partial^2 H_x}{\partial y^2}\right]v-H_zH_xw_{xx}-H_yH_zw_{xy}+\left[H_xH_y-H_z\frac{\partial H_x}{\partial x}+ikH_y^2+ikS^2\right]w_x$$

$$+\left[H_y^2-H_z\frac{\partial H_y}{\partial x}-ikH_yH_x\right]w_y+\left[ikH_yH_z+k^2H_xH_z+ikH_y\frac{\partial H_y}{\partial x}\right.$$

$$\left.-ikH_y\frac{\partial H_x}{\partial y}\right]w=0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.1)$$

$$-H_xH_yu_{xx}+\left(S^2+H_z^2\right)u_{xy}-H_xH_yu_{yy}+\left[H_xH_z-2H_x\frac{\partial H_y}{\partial x}+H_z\frac{\partial H_z}{\partial y}-ikH_yH_z\right]u_x$$

$$-\left[H_x\left(ikH_z-\frac{\partial H_x}{\partial x}\right)+H_x\frac{\partial H_y}{\partial y}\right]u_y+\left[ikH_z^2+H_x\frac{\partial H_z}{\partial x}-H_x\frac{\partial^2 H_y}{\partial x^2}\right.$$

$$\left.-H_x\left(ik\frac{\partial H_z}{\partial y}-\frac{\partial^2 H_x}{\partial x\partial y}\right)-ikH_z\frac{\partial H_y}{\partial x}\right]u+H_x^2v_{xx}+(S^2+H_x^2+H_z^2)v_{yy}$$

$$+\left[H_x\left(ikH_z-\frac{\partial H_y}{\partial y}\right)+H_x\frac{\partial H_x}{\partial x}+ikH_xH_z\right]v_x+2\left(H_x\frac{\partial H_x}{\partial y}+H_z\frac{\partial H_z}{\partial y}\right)v_y$$

$$+\left[\omega^2-H_z\frac{\partial H_x}{\partial y}+H_x\frac{\partial H_z}{\partial y}+H_x\left(ik\frac{\partial H_z}{\partial x}-\frac{\partial^2 H_y}{\partial x\partial y}\right)+H_x\frac{\partial^2 H_x}{\partial y^2}+H_z\frac{\partial^2 H_z}{\partial y^2}\right.$$

$$\left.+ikH_z\left(ikH_z-\frac{\partial H_y}{\partial y}\right)\right]v-H_xH_zw_{xy}-H_zH_yw_{yy}-\left[H_x^2+ikH_xH_y+H_z\frac{\partial H_x}{\partial y}\right]w_x$$

$$+\left[ikS^2-H_xH_y+ikH_x^2-H_z\frac{\partial H_y}{\partial y}\right]w_y+\left[-ikH_xH_z-ikH_x\frac{\partial H_y}{\partial x}+ikH_x\frac{\partial H_x}{\partial y}\right.$$

$$\left.+k^2H_yH_z\right]w=0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.2)$$

$$-H_xH_zu_{xx}-H_yH_zu_{xy}+\left[ikS^2-H_xH_y-H_y\frac{\partial H_z}{\partial y}+ikH_y^2-2H_x\frac{\partial H_z}{\partial x}\right]u_x$$

$$-ikH_xH_yu_y+\left[-H_x\frac{\partial H_y}{\partial x}-H_y\left(ikH_z-\frac{\partial H_x}{\partial x}\right)-H_y\frac{\partial^2H_z}{\partial x\partial y}+ikH_y\frac{\partial H_y}{\partial x}\right.$$

$$\left.-ikH_x\left(ikH_z-\frac{\partial H_x}{\partial x}\right)-H_x\frac{\partial^2H_z}{\partial x^2}\right]u-H_yH_zv_{yy}-H_xH_zv_{xy}+\left[H_x^2-ikH_xH_y\right.$$

$$\left.-H_x\frac{\partial H_z}{\partial y}\right]v_x+\left[ikS^2+H_xH_y-2H_y\frac{\partial H_z}{\partial y}+ikH_x^2-H_x\frac{\partial H_z}{\partial x}\right]v_y+\left[H_x\left(ikH_z-\frac{\partial H_y}{\partial y}\right)\right.$$

$$\left.+H_y\frac{\partial H_x}{\partial y}-H_y\frac{\partial^2H_z}{\partial y^2}-ikH_y\left(ikH_z-\frac{\partial H_y}{\partial y}\right)+ikH_x\frac{\partial H_x}{\partial y}-H_x\frac{\partial^2H_z}{\partial x\partial y}\right]v+H_x^2w_{xx}$$

$$+2H_xH_yw_{xy}+H_y^2w_{yy}+\left[H_y\frac{\partial H_x}{\partial y}+H_x\frac{\partial H_x}{\partial x}\right]w_x+\left[H_y\frac{\partial H_y}{\partial y}+H_x\frac{\partial H_y}{\partial x}\right]w_y$$

$$+\left[\omega^2-k^2S^2-k^2H_y^2-k^2H_x^2\right]w=0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.3)$$

where H_x, H_y, H_z are given by

$$H_x=-b\cos ax\sin by,\quad H_y=a\sin ax\cos by,\quad H_z=\cos ax\cos by \qquad (3.4)$$

with $a=\kappa/\alpha$, $b=l/\alpha$ and $a^2+b^2=1$.

As we shall limit ourselves to the neighbourhood of neutral lines, we put

$$x=\frac{\pi}{2a}+\xi,\quad y=\frac{\pi}{2b}+\eta \qquad .. \quad .. \quad .. \quad (3.5)$$

where ξ and η are small. We then have

$$\left.\begin{aligned}H_x&=ab\xi-\frac{ab^3}{2}\xi\eta^2-\frac{a^3b}{6}\xi^3+\ldots\\H_y&=-ab\eta+\frac{a^3b}{2}\xi^2\eta+\frac{ab^3}{6}\eta^3+\ldots\\H_z&=ab\xi\eta-\frac{a^3b}{6}\xi^3\eta-\frac{ab^3}{6}\xi\eta^3+\ldots\end{aligned}\right\} \quad .. \quad .. \quad (3.6)$$

Substituting for H_x, H_y and H_z in (3.1), (3.2), (3.3) and retaining up to quadratic terms in ξ and η we get

$$(C_s^2+\eta^2)u_{\xi\xi}+\eta^2u_{\eta\eta}+2\eta u_\eta+(W^2-2ik\xi\eta)u+\xi\eta v_{\xi\xi}$$
$$+C_s^2v_{\xi\eta}+\xi\eta v_{\eta\eta}+2\eta v_\xi+ik\eta^2v+(ik\eta^2-2\xi\eta+ikC_s^2)w_\xi$$
$$+(\eta^2+ik\xi\eta)w_\eta=0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3.7)$$

$$\xi\eta u_{\xi\xi}+C_s^2u_{\xi\eta}+\xi\eta u_{\eta\eta}+2\xi u_\eta-ik\xi^2u$$
$$+\xi^2v_{\xi\xi}+(C_s^2+\xi^2)v_{\eta\eta}+2\xi v_\xi+(W^2+2ik\xi\eta)v$$
$$+(ik\xi\eta-\xi^2)w_\xi+(ikC_s^2+2\xi\eta+ik\xi^2)w_\eta=0 \quad .. \quad .. \quad (3.8)$$

$$ik(C_s^2+\eta^2)u_\xi+ik\xi\eta u_\eta+ik\xi u+ik\xi\eta v_\xi$$
$$+(C_s^2+\xi^2)ikv_\eta+ik\eta v+\xi^2 w_{\xi\xi}-2\xi\eta w_{\xi\eta}$$
$$+\eta^2 w_{\eta\eta}+\xi w_\xi+\eta w_\eta+(W^2-k^2C_s^2-k^2\xi^2-k^2\eta^2)w=0 \qquad (3.9)$$

where

$$C_s^2=\frac{S^2}{a^2b^2}, \quad W^2=\frac{\omega^2}{a^2b^2}$$

Consistent with our approximation we assume

$$\left.\begin{aligned} u &= a_0+a_1\xi+a_2\eta+a_3\xi^2+a_4\xi\eta+a_5\eta^2+\dots \\ v &= b_0+b_1\xi+b_2\eta+b_3\xi^2+b_4\xi\eta+b_5\eta^2+\dots \\ w &= c_0+c_1\xi+c_2\eta+c_3\xi^2+c_4\xi\eta+c_5\eta^2+\dots \end{aligned}\right\} \quad \dots \quad (3.10)$$

Substituting the above expressions for u, v and w in eqns. (3.7)–(3.9) and equating to zero the coefficients of $\xi^r\eta^s$, $0 \leqslant r+s \leqslant 2$, we obtain the following linear homogeneous equations determining a's, b's and c's:

$$2C_s^2a_3+W^2a_0+C_s^2b_4+ikC_s^2c_1=0 \quad \dots \quad (3.11)$$
$$W^2a_1+2ikC_s^2c_3=0 \quad \dots \quad (3.12)$$
$$(W^2+2)a_2+2b_1+ikC_s^2c_4=0 \quad \dots \quad (3.13)$$
$$W^2a_3=0 \quad \dots \quad (3.14)$$
$$(W^2+2)a_4-2ika_0+6b_3+2b_5-2c_1+ikc_2=0 \quad \dots \quad (3.15)$$
$$2a_3+(W^2+6)a_5+2b_4+ikb_0+ikc_1+c_2=0 \quad \dots \quad (3.16)$$
$$C_s^2a_4+2C_s^2b_5+W^2b_0+ikC_s^2c_2=0 \quad \dots \quad (3.17)$$
$$2a_2+(W^2+2)b_1+ikC_s^2c_4=0 \quad \dots \quad (3.18)$$
$$W^2b_2+2ikC_s^2c_5=0 \quad \dots \quad (3.19)$$
$$2a_4-ika_0+(W^2+6)b_3+2b_5-c_1+ikc_2=0 \quad \dots \quad (3.20)$$
$$2a_3+6a_5+(W^2+2)b_4+2ikb_0+ikc_1+2c_2=0 \quad \dots \quad (3.21)$$
$$W^2b_5=0 \quad \dots \quad (3.22)$$
$$ikC_s^2a_1+ikC_s^2b_2+(W^2-k^2C_s^2)c_0=0 \quad \dots \quad (3.23)$$
$$2ikC_s^2a_3+ika_0+ikC_s^2b_4+(W^2-k^2C_s^2+1)c_1=0 \quad \dots \quad (3.24)$$
$$ikC_s^2a_4+2ikC_s^2b_5+ikb_0+(W^2-k^2C_s^2+1)c_2=0 \quad \dots \quad (3.25)$$
$$ika_1+ikb_2+(W^2-k^2C_s^2+4)c_3-k^2c_0=0 \quad \dots \quad (3.26)$$
$$2ika_2+2ikb_1+(W^2-k^2C_s^2)c_4=0 \quad \dots \quad (3.27)$$
$$ika_1+ikb_2+(W^2-k^2C_s^2+4)c_5-k^2c_0=0 \quad \dots \quad (3.28)$$

Equations (3.14) and (3.22) give $a_3=0$ and $b_5=0$. We are left with 16 unknowns and 16 equations. These form three mutually exclusive sets, say,

A, B and C. The set A consisting of the three equations (3.13), (3.18), (3.27) in the unknowns a_2, b_1 and c_4; the set B consisting of the five equations (3.12), (3.19), (3.23), (3.26), (3.28) in the unknowns a_1, b_2, c_0, c_3 and c_5, and the set C consisting the eight equations (3.11), (3.15), (3.16), (3.17), (3.20), (3.21), (3.24), (3.25) in the unknowns a_0, a_4, a_5, b_0, b_3, b_4, c_1 and c_2.

Each of the three sets leads to dispersion relation when we apply the condition for the non-trivial solution of the equations.

The set A gives the dispersion relation

$$W^2 = k^2C_s^2 - 4$$

or in physical parameters

$$\frac{\omega^2}{k^2} = c_s^2 - \frac{4\kappa^2 l^2}{\alpha^2 k^2} V_A^2 \quad .. \quad .. \quad .. \quad .. \quad (3.29)$$

This relation represents the sonic modes modified by the presence of inhomogeneous magnetic field. These modes are stable or unstable according as

$$k \gtrless \frac{2\kappa l}{\alpha} \frac{V_A}{c_s}.$$

The set B breaks up into the following two relations:

$$\text{(i)} \quad W^2 - k^2C_s^2 + 4 = 0 \quad .. \quad .. \quad .. \quad .. \quad (3.30)$$

$$\text{(ii)} \quad W^4 + 2(2 - k^2C_s^2)W^2 + k^4C_s^4 = 0 \quad .. \quad .. \quad .. \quad (3.31)$$

The first relation is the same as obtained in set A, while the second gives on solving

$$\omega^2 = \left(k^2c_s^2 - \frac{2\kappa^2 l^2}{\alpha^2} V_A^2\right) \pm \frac{2\kappa l}{\alpha} V_A \sqrt{\frac{\kappa^2 l^2}{\alpha^2} V_A^2 - k^2 c_s^2} \quad .. \quad .. \quad (3.32)$$

Now, if $k^2 < \frac{\kappa^2 l^2}{\alpha^2} \frac{V_A^2}{c_s^2}$, the two values of ω^2 are real and negative, giving rise to growing and decaying modes while if $k^2 > \frac{\kappa^2 l^2}{\alpha^2} \frac{V_A^2}{c_s^2}$, the values of ω^2 are complex conjugate, again leading to instability.

Finally, the set C gives the dispersion relation which breaks up into the following two relations:

$$(W^2+7)[W^4 + (1 - k^2C_s^2)W^2 + k^2C_s^2] + W^2(W^2+1) \pm 4kC_s^2(W^2+3) = 0 \quad (3.33)$$

We can easily show that each of these factors, cubic on W^2 possesses at least one negative root < -3, thus giving growing and decaying modes.

From the above discussion, we conclude that there is always instability in the neighbourhood of neutral lines. This result is in agreement with the conclusion drawn by Schatzman that neutral lines contribute to instability. However, by employing normal mode technique, we are able to obtain magnitudes of growth rates of instability present around neutral lines.

4. Study of Stability through Energy Method

In this section we again investigate the stability of the system (1.1) on the basis of energy principle given by Bernstein *et al.* (1958) and also employed by Schatzman.

The energy principle states that if

$$\delta W = -\tfrac{1}{2}\int \vec{\xi} \cdot \vec{F}\left(\vec{\xi}\right) d\tau \qquad .. \qquad .. \qquad .. \quad (4.1)$$

where $\vec{\xi}$ denotes an arbitrary displacement from the equilibrium,

$$\vec{F}\left(\vec{\xi}\right) = \gamma p_0 \operatorname{grad}\left(\operatorname{div}\vec{\xi}\right) + \left(\operatorname{curl}\vec{B}_0\right)\times\vec{Q} - \vec{B}_0 \times \operatorname{curl}\vec{Q},$$

$$\vec{Q} = \operatorname{curl}\left(\vec{\xi}\times\vec{B}_0\right)$$

and the integration is performed over the smallest cell bounded by the neutral lines, we have stability or instability according as $\delta W > \text{or} < 0$.

Following Schatzman, we assume

$$\vec{\xi} = \sum_m \sum_n (a_{mn}, b_{mn}, c_{mn}) \exp i(m\kappa x + nly + rz) \qquad .. \qquad .. \quad (4.2)$$

The integral (4.1) reduces to

$$-\frac{K}{2}\sum_m \sum_n \left[\vec{\xi}_{mn} \cdot \vec{F}\left(\vec{\xi}^*_{mn}\right) + \vec{\xi}^*_{mn} \cdot \vec{F}\left(\vec{\xi}_{mn}\right)\right] \qquad .. \qquad .. \quad (4.3)$$

where K denotes a positive constant.

If $\vec{R} = \vec{\xi} \times \vec{B}_0$, the contribution to (δW) from magnetic part is

$$\tfrac{1}{2}\left\{\vec{R}^* \cdot \left[\alpha \operatorname{curl}\vec{R} - \operatorname{curl}\operatorname{curl}\vec{R}\right] + \vec{R} \cdot \left[\alpha \operatorname{curl}\vec{R}^* - \operatorname{curl}\operatorname{curl}\vec{R}^*\right]\right\} \qquad .. \quad (4.4)$$

Taking $$\vec{R}_{mn} = (A_{mn}, B_{mn}, C_{mn}) \exp i(m\kappa x + nly + rz) \qquad .. \quad (4.5)$$

the m, nth component of (4.4) reduces to, after dropping subscripts from A, B, C, A^*, B^* and C^*,

$$\begin{aligned} &mn\kappa l(AB^* + A^*B) + nlr(BC^* + B^*C) + m\kappa r(AC^* + A^*C) \\ &-(l^2n^2 + r^2)AA^* - (m^2\kappa^2 + r^2)BB^* - (m^2\kappa^2 + n^2l^2)CC^* \\ &+\alpha i[r(AB^* - A^*B) + m\kappa(BC^* - B^*C) + nl(CA^* - C^*A)] \qquad .. \quad (4.6)\dagger \end{aligned}$$

We can express the components of $\vec{R}$ in terms of the components of $\vec{\xi}$ to obtain A, B, C as follows:

$$\begin{aligned} A_{mn} = \frac{H_0}{4}\Bigg[&(b_{m-1,\,n-1} + b_{m-1,\,n+1} + b_{m+1,\,n-1} + b_{m+1,\,n+1}) \\ &-\frac{\kappa}{\alpha i}(c_{m-1,\,n-1} + c_{m-1,\,n+1} - c_{m+1,\,n-1} - c_{m+1,\,n+1})\Bigg] \qquad .. \quad (4.7) \end{aligned}$$

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† This expression differs from that of Schatzman. Perhaps some error has crept in his work due to some slip.

$$B_{mn} = -\frac{H_0}{4}\left[\frac{l}{\alpha i}(c_{m-1,\,n-1}-c_{m-1,\,n+1}+c_{m+1,\,n-1}-c_{m+1,\,n+1})\right.$$
$$\left.+(a_{m-1,\,n-1}+a_{m-1,\,n+1}+a_{m+1,\,n-1}+a_{m+1,\,n+1})\right] \quad .. \quad (4.8)$$

$$C_{mn} = \frac{H_0}{4\alpha i}[\kappa(a_{m-1,\,n-1}+a_{m-1,\,n+1}-a_{m+1,\,n-1}-a_{m+1,\,n+1})$$
$$+l(b_{m-1,\,n-1}-b_{m-1,\,n+1}+b_{m+1,\,n-1}-b_{m+1,\,n+1})] \quad .. \quad (4.9)$$

Substituting for A, B, ... in terms of a, b, ... and collecting the terms with index mn, we finally obtain the following expression for (δW):

$$\frac{4}{KH_0^2}(\delta W)_{mn} = \frac{C_s^2}{V_A^2}|m\kappa a+nlb+rc|^2+\frac{l^2}{\alpha^2}(r^2+n^2\kappa^2)|a|^2$$
$$+\frac{\kappa^2}{\alpha^2}(r^2+m^2l^2)|b|^2+\left[r^2+(m^2+n^2)\frac{\kappa^2l^2}{\alpha^2}\right]|c|^2$$
$$+|m\kappa a+nlb|^2+\frac{\kappa^2}{\alpha^2}|rc+ma|^2+\frac{l^2}{\alpha^2}|rc+nb|^2 \quad .. \quad (4.10)$$

where C_s, V_A are the sound speed and Alfvén speed respectively. As the expression (4.10) shows (δW) is positive definite and, therefore, the system as a whole is stable. This is in contradiction to the result obtained by Schatzman where he has shown that instabilities are present in z-direction if

$$r^2 < \frac{\alpha^2 B_0^2}{32\pi}\frac{\sin^2 2\omega}{\gamma p_0}, \quad \omega = \tan^{-1}\frac{l}{\kappa}.$$

5. Conclusion

By the energy principle which in a way provides a gross treatment, every cell appears to be stable but the contribution to δW by the neighbourhood of a neutral line is negative as proved by Schatzman or by the present investigation in § 3. This explains the motivation of the present investigation. Moreover, the present study provides the growth rate of the instabilities.

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References

Bernstein, I. B., Frieman, E. A., Kruskal, M. D., and Kulsrud, R. M. (1958). An energy principle for hydromagnetic stability problems. *Proc. R. Soc.*, 244 A, 17.

Bhatnagar, P. L. (1957). Lectures on Magnetohydrodynamics, Indian Institute of Science, Bangalore.

Chakraborty, B. B., and Bhatnagar, P. L. (1960). The stability of force-free magnetic fields. *Proc. natn. Inst. Sci. India*, A 26, 592.

Chandrasekhar, S. (1956). On force-free magnetic fields. *Proc. natn. Acad. Sci.*, **42**, 1.

Lüst, R., and Schlüter, A. (1954). Kraftfreie Magnetfelder. *Z. Astrophys.*, **34**, 263.

Menzel, D. H. (1964). Magnetohydrodynamic flow and force-free fields. *Astrophys. norv.*, **9**, 225.

Schatzman (1960). On the stability of a periodic force-free field with neutral lines. *Rev. mod. Phys.*, **32**, 903.

Trehan, S. K. (1957). On the stability of force-free magnetic fields. *Astrophys. J.*, **126**, 429.

Woltjer, L. (1958). The stability of force-free magnetic fields. *Astrophys. J.*, **128**, 384.